# Large scale geometry of curve complexes

by

**Kate Mackintosh Vokes**

**Thesis**

Submitted to the University of Warwick

for the degree of

**Doctor of Philosophy**

**Department of Mathematics**

May 2018

# Contents

# List of Figures

# Acknowledgments

The work of this thesis would not have been possible without the guidance and contribution of my supervisor, Brian Bowditch. I am grateful to him for many interesting conversations, valuable suggestions and comments on written work. I would also particularly like to thank Francesca Iezzi, Beatrice Pozzetti and Saul Schleimer for interesting conversations and friendly advice. I would like to thank Saul Schleimer and Henry Wilton for their very helpful comments on this thesis.

I am indebted to the Mathematics department at Warwick for providing a welcoming and stimulating environment, and to the Geometry and Topology group in particular. I am particularly grateful for the friends I have made here who have supported and encouraged me. The funding that enabled me to undertake my PhD was provided by the Engineering and Physical Sciences Research Council.

Finally, I would like to thank my family for being there for me throughout and for taking an interest in the most abstract of subjects.

# Declarations

The work of this thesis is my own, except where otherwise indicated in the text, or where the work is widely known. No part has been submitted by me for any other degree.

The work of Chapter 4 and Section 5.2 has appeared in preprints ([58] and [57] respectively), and has been, or will be, submitted for publication.

All illustrations were created by the author using Inkscape.

# Abstract

We study the coarse geometry of curve graphs and related graphs for connected, compact, orientable surfaces.

We prove that the separating curve graph of a surface is a hierarchically hyperbolic space, as defined by Behrstock, Hagen and Sisto, whenever it is connected. It also automatically has the coarse median property defined by Bowditch. Consequences for the separating curve graph include a distance formula analogous to Masur and Minsky's distance formula for the mapping class group, an upper bound on the maximal dimension of quasiflats, and the existence of a quadratic isoperimetric inequality.

We also describe surgery arguments for studying the coarse geometry of curve graphs and similar graphs. Specifically, we give a new proof of the uniform hyperbolicity of the curve graphs, extending methods of Przytycki and Sisto. We also give an elementary proof of Masur and Minsky's result that the disc graphs are quasiconvex in the curve graphs. Moreover, we show that the constant of quasiconvexity is independent of the surface, as also shown in work of Hamenstädt.

# Abbreviations

### Standard notations

| | |
|---|---|
| $\mathrm{int}(X)$ | The interior of $X$ |
| $\bar{X}$ | The closure of $X$ |
| $\partial X$ | The boundary of $X$ |
| $\mathrm{diam}_X(Y)$ | The diameter of a subset $Y$ of $X$ |
| $N_X(Y, K)$ | The $K$-neighbourhood of $Y$ in $X$ |
| $2^{\mathcal{S}}$ | The power set of $\mathcal{S}$ |
| $H_k(X; \mathbb{Z})$ | The $k$-th homology group of $X$, with integer coefficients |
| $H_k(X, A; \mathbb{Z})$ | The $k$-th homology group of $X$ relative to $A \subseteq X$, with integer coefficients |
| $S^1$ | The circle, identified with $\{z \in \mathbb{C} \mid \lvert z\rvert = 1\}$ |

### Notations defined in the text (with page references)

| | | | | | |
|---|---|---|---|---|---|
| $\asymp_{K_1,K_2}$ | 5 | $\xi(S)$ | 11 | $\mathrm{Sep}(S)$ | 13 |
| $d_H(A, B)$ | 7 | $\mathrm{MCG}(S)$ | 11 | $\mathcal{D}(M, S)$ | 14 |
| $S_{g,b}$, $S_g$ | 9 | $\mathrm{Teich}(S)$ | 12 | $\partial_S X$ | 14 |
| $i(\alpha, \beta)$ | 10 | $\mathcal{C}(S)$ | 13 | $[x]_C$ | 16 |

**Conventions** We make the following assumptions and abbreviations (see the pages given).

1. Metric spaces will be geodesic metric spaces. 6
2. Surfaces will be connected, oriented and compact. 9
3. Curves will be essential, non-peripheral simple closed curves. 9
4. Curves will often be considered up to isotopy. 10
5. We abbreviate $d_{\mathcal{C}(S)}$ to $d_S$. 13
6. Subsurfaces will be isotopy classes of essential subsurfaces. 14
7. When considering maps between curve graphs, these will really be maps between their vertex sets, and not, in general, graph morphisms. 14
8. We define the distance between finite sets of vertices in a graph to be the diameter of their union. 15
9. When considering the distance between subsurface projections, we often do not write the projection maps. 15
10. For a subsurface $X$ of a surface $S$, we write $S \setminus X$ for the closure of $S \setminus X$. 25
11. For a multicurve $a$ in a surface $S$, we write $S \setminus a$ to refer to removing a regular open neighbourhood of $a$ from $S$. 25

# Chapter 1

# Introduction

In recent years, certain combinatorial objects associated to surfaces have become invaluable in studying mapping class groups and Teichmüller spaces of surfaces, with wider applications to the geometry of 3-manifolds. The *curve complex* of a surface, introduced by Harvey [32] as an analogue of Tits buildings for mapping class groups and Teichmüller spaces, has a vertex for each isotopy class of curves in the surface and a $k$-simplex for each set of $k + 1$ disjoint curves. It was early on applied by Harer to study homological properties of the mapping class groups [30, 31]. Two substantial works by Masur and Minsky [41, 42] used the curve complex to study the large scale geometry of Teichmüller space and the mapping class group, and led on to much other work on this theme. The curve complex also played a crucial part in the proof of Thurston's Ending Lamination Conjecture by Minsky [46] and Brock, Canary and Minsky [19]. This is a rigidity result stating that a complete hyperbolic 3-manifold with finitely generated fundamental group is determined by its topology and certain *end invariants*. Since the curve complex is a flag complex, all combinatorial information is encoded in its 1-skeleton, the *curve graph*, and that is what we shall always consider here.

Many variations on the curve graph have also been defined, each giving slightly different information. For example, the *marking graph* used in [42] is quasi-isometric to the mapping class group, and the *pants graph* was shown by Brock to be quasi-isometric to the Weil–Petersson metric on Teichmüller space, with applications to the geometry of quasifuchsian 3-manifolds [18].

Masur and Minsky showed in [41] that the curve graph of any surface is hyperbolic in the sense of Gromov (and has infinite diameter, except for a few trivial examples). Neither the mapping class group nor the Teichmüller space is hyperbolic, but it had been observed that both have some hyperbolic-like behaviour. This was made more precise by Masur and Minsky in [41] and [42], and later axiomatised by Behrstock, Hagen and Sisto in the theory of *hierarchically hyperbolic spaces* [6, 7]. One of the objectives of this thesis is to make progress towards bringing a large class of objects associated to surfaces into a general framework by showing that they are hierarchically hyperbolic spaces. Our current contribution to this is to show that the *separating curve graph* is a hierarchically hyperbolic space (Chapter 4). However, we suggest that the methods of this chapter may be more generally applicable.

Another topic of this thesis is the use of surgery arguments to investigate the large scale geometry of curve graphs and other such graphs. One benefit of such methods is that often the argument is very explicit and does not depend on the specific surface we are working with. The proof of the hyperbolicity of the curve graph by Masur and Minsky in [41] does not give an estimate for the constant of hyperbolicity, and, in particular, this constant *a priori* depends on the surface. However, it was proved independently by Aougab [1], Bowditch [12], Clay, Rafi and Schleimer [20] and Hensel, Przytycki and Webb [35] that the curve graphs are *uniformly hyperbolic*, that is, there is a single constant of hyperbolicity which applies for all surfaces. Surgery methods are central to [20] and [35], and [35] obtained a particularly small constant.

Inspired by the *unicorn arcs* introduced by Hensel, Przytycki and Webb in [35], Przytycki and Sisto gave a new proof of the uniform hyperbolicity of the curve graphs of closed surfaces using *bicorn curves* [48]. In this thesis (Section 5.1), we extend the methods of [48] to also apply to surfaces with boundary. Bicorn curves have also been applied by A. Rasmussen to give a proof of the uniform hyperbolicity of the non-separating curve graphs (including for surfaces with boundary) [50]. Also making use of the results of [48], we give an elementary proof of the uniform quasiconvexity of the disc graphs in the curve graphs. The quasiconvexity of the disc graphs in the curve graphs was proved by Masur and Minsky [43], with constants there depending on the surface. Masur and Minsky's proof uses a study of *train tracks* on surfaces. A result of Hamenstädt on train track splitting sequences (Section 3 of [29]) implies that the constant of quasiconvexity can be taken to be independent of the surface. Our proof uses disc surgeries described in [43] but bypasses the use of train tracks by using results on bicorn curves. The way in which the disc graph sits inside the curve graph is of interest in part because of its applications to *Heegaard splittings* of 3-manifolds. A Heegaard splitting where two handlebodies are glued along a surface $S$ can be specified by curves in $S$ which bound discs in one or other of the handlebodies. The disc graphs of the two handlebodies sit inside the curve graph of $S$, and the *Hempel distance* for a Heegaard splitting [34] is defined to be the minimal distance between a vertex in one disc graph and a vertex in the other. Hempel studied how this distance affects the topology of the 3-manifold. The quasiconvexity of the disc graph in the curve graph was used by Masur and Schleimer in giving a method of coarsely computing the Hempel distance [45]. Another application of the disc graphs is to the study of handlebody groups (that is, mapping class groups of handlebodies).

## 1.1 Overview of content

Chapters 2 and 3 are expository. Chapter 2 introduces concepts in coarse geometry, including definitions and standard results that we shall use later. Chapter 3 gives background on curves in surfaces and introduces many of the objects we shall be considering in this thesis, such as the mapping class group, Teichmüller space, the curve graph and variations. We also give definitions of the coarse median property and hierarchical hyperbolicity, together with some consequences.

In Chapter 4 we prove that the separating curve graph is a hierarchically hyperbolic space whenever it is connected. In Section 4.1.3, we give a proof of connectedness of the separating curve graph whenever this holds. This is a well known result for which we were unable to find a proof in the literature. In Section 4.2, we introduce a new graph, $\mathcal{K}(S)$, which we prove in Section 4.3 to be quasi-isometric to the separating curve graph. We prove in Sections 4.2.2 and 4.2.3 that $\mathcal{K}(S)$ satisfies the definition of hierarchical hyperbolicity, and the quasi-isometry invariance of hierarchical hyperbolicity [6] then implies that the separating curve graph is hierarchically hyperbolic. Proving that $\mathcal{K}(S)$ is hierarchically hyperbolic involves verifying the nine axioms for hierarchical hyperbolicity set out by Behrstock, Hagen and Sisto in [7]. The most substantial part of the proof is the verification of Axiom 9, which we give as Proposition 4.2.4.

Chapter 5 investigates surgery arguments. In Section 5.1, we give a new proof of the uniform hyperbolicity of the curve graphs, based on methods of Przytycki and Sisto [48], but applying to surfaces with boundary as well as closed surfaces. The method is to define a subgraph of the curve graph associated to each pair of curves, $\alpha$, $\beta$, by including precisely those curves which can be formed from $\alpha$ and $\beta$ by certain surgeries. We show that these subgraphs satisfy a criterion for hyperbolicity (Proposition 5.1.2 here) due to Masur–Schleimer [45] and Bowditch [12], and related to previous work of Gilman [27]. In particular we show that for any triple of curves, the triangle given by these subgraphs is “slim”. The constants involved are independent of the surface.

In Section 5.2, we give an elementary proof that the disc graph associated to a boundary component $S$ of a 3-manifold $M$ is $K$-quasiconvex in the curve graph of $S$, with constant $K$ independent of $S$ and $M$. We again use Proposition 5.1.2, this time observing that the disc surgeries described by Masur and Minsky in their proof of quasiconvexity [43] give vertices of the curve graph which lie inside a set satisfying the hypotheses of the proposition. For the purposes of this section, the important consequence of Proposition 5.1.2 is that for a subgraph $\mathcal{L}(\alpha, \beta)$ associated to curves $\alpha$, $\beta$ and satisfying the hypotheses, any geodesic between $\alpha$ and $\beta$ in the curve graph stays at a bounded Hausdorff distance from $\mathcal{L}(\alpha, \beta)$. We use some standard arguments to show that this implies that any geodesic in $\mathcal{C}(S)$ joining two curves which bound discs stays at a uniformly bounded distance from the disc graph. Again, any constants are independent of the surface.

# Chapter 2

# Coarse geometry

In this chapter, we give some definitions and state some known results in coarse geometry. References for the material of this chapter include [10, 17, 21].

## 2.1 Gromov hyperbolicity and other definitions

Many of the ideas in coarse geometry stem from work of Gromov [28]. A key application is to the study of the geometry of groups (see Section 2.3). The informal idea of *coarse*, or *large scale*, geometry is that we can suppose that we look at each space from far away, so that small changes of distance become negligible. This is made precise by the notion of quasi-isometry.

**Definition 2.1.1.** Let $(X, d_X)$ and $(Y, d_Y)$ be metric spaces, and $A, B \in \mathbb{R}$.

1. Let $K_1 \geqslant 1$ and $K_2 \geqslant 0$. We write $A \asymp_{K_1,K_2} B$ if $\frac{1}{K_1}(A - K_2) \leqslant B \leqslant K_1 A + K_2$.

2. A (not necessarily continuous) function $\phi : X \to Y$ is a $(K_1, K_2)$-*quasi-isometric embedding* if there exist constants $K_1 \geqslant 1$ and $K_2 \geqslant 0$ such that, for any $x_1, x_2 \in X$, we have $d_X(x_1, x_2) \asymp_{K_1,K_2} d_Y(\phi(x_1), \phi(x_2))$.

3. The map $\phi$ is a *quasi-isometry* if, in addition, there exists $k_3 \geqslant 0$ such that, for any $y \in Y$, there is some $x \in X$ satisfying $d_Y(y, \phi(x)) \leqslant k_3$.

4. If there exists a quasi-isometry $\phi : X \to Y$ then $X$ and $Y$ are *quasi-isometric*.

If a function satisfies the upper bound for a quasi-isometric embedding without necessarily satisfying the lower bound, then it is $(K_1, K_2)$-*coarsely Lipschitz*. If $K_2 = 0$, then the function is $K_1$-*Lipschitz*.

The property of $\delta$-*hyperbolicity*, *Gromov hyperbolicity*, or simply *hyperbolicity*, is a concept of negative curvature which can be applied to general metric spaces, unlike more traditional notions of curvature in differential geometry. There are a number of equivalent definitions. We will use one of the most common, phrased in terms of "$\delta$-slim triangles".

Firstly, recall that a *geodesic* between two points in a metric space $(X, d_X)$ is a path $\gamma : I \to X$, for some interval $I$, such that for any $t, u \in I$, we have $d_X(\gamma(t), \gamma(u)) = |t - u|$. The metric space $X$ is a *geodesic metric space* if for any two points $a, b \in X$, there exists some geodesic in $X$ joining $a$ and $b$. From now on, all metric spaces will be geodesic spaces unless stated otherwise. It is possible to formulate a definition of hyperbolicity without assuming this property, but that will not be necessary here.

**Definition 2.1.2.** Suppose $\delta \geqslant 0$. A geodesic metric space $X$ is *$\delta$-hyperbolic* if every triangle in $X$ whose three edges are geodesics has the property that the closed $\delta$-neighbourhood of any two of the sides contains the third side.

The constant $\delta$ is the *constant of hyperbolicity*. It is not unique; in particular, any larger constant will also work.

**Proposition 2.1.3.** *If $X$ and $Y$ are quasi-isometric, then $Y$ is hyperbolic if and only if $X$ is. Moreover, the constant of hyperbolicity of $Y$ depends only on that of $X$ and on the quasi-isometry constants.* $\square$

See, for example, Theorem III.H.1.9 of [17] for a proof. Properties such as hyperbolicity which are invariant under quasi-isometries are sometimes called *large scale* properties.

**Definition 2.1.4.** A subset $Y$ of a metric space $X$ is *$K$-quasiconvex* in $X$ if for any two points, $y$ and $y'$, in $Y$, any geodesic in $X$ joining $y$ and $y'$ is contained within the closed $K$-neighbourhood of $Y$ in $X$.

This generalises the notion of convex subsets. We will denote the closed $K$-neighbourhood of $Y$ in $X$ by $N_X(Y, K)$.

## 2.2 Properties of hyperbolic spaces

As mentioned above, there are a number of characterisations of hyperbolicity equivalent to Definition 2.1.2; see, for example, Chapter 1 of [21] for a discussion. One common definition is that a space $X$ is hyperbolic if there exists $k$ such that any geodesic triangle in $X$ has a *$k$-centre*, that is, a point that is at most distance $k$ from some point on each edge of the triangle. For a space $X$ that is $\delta$-hyperbolic as in Definition 2.1.2, such a constant $k$ is bounded in terms of $\delta$. Moreover, given three points in $X$, the set of points which are $k$-centres for geodesic triangles with these three points as vertices has diameter bounded above in terms of $k$. Hence, we can think of choosing a $k$-centre of a triangle as a coarsely well defined ternary operation on $X$, a concept to which we shall return in Section 3.6.

**Definition 2.2.1.** A $(\lambda, h)$-*quasigeodesic* in a metric space $X$ is a $(\lambda, h)$-quasi-isometric embedding $\gamma\colon I \to X$ (or $\gamma\colon I \cap \mathbb{Z} \to X$), where $I$ is an interval of $\mathbb{R}$.

In non-hyperbolic metric spaces, such as Euclidean space, a quasigeodesic need not be close to any actual geodesic. However, in a hyperbolic space, quasi-geodesics do stay close to geodesics. We state this result, sometimes referred to as the Morse Lemma, more precisely below in Proposition 2.2.3, after recalling the definition of Hausdorff distance. See, for example, Proposition 6.17 of [10] for a proof (or, for infinite quasigeodesics, Theorem 3.3.1 of [21]).

**Definition 2.2.2.** The *Hausdorff distance* between two subsets $A$ and $B$ of a metric space $X$ is

$$d_H(A, B) = \inf\{r \in [0, \infty] \mid A \subseteq N_X(B, r), B \subseteq N_X(A, r)\}.$$

**Proposition 2.2.3.** *Let $X$ be a $\delta$-hyperbolic space. Let $\alpha$ be a geodesic in $X$ and $\beta$ a $(\lambda, h)$-quasigeodesic with the same endpoints. Then the Hausdorff distance between $\alpha$ and $\beta$ is bounded above by a constant depending on $\delta$, $\lambda$ and $h$.* □

Another feature of hyperbolic spaces is their "tree-like" nature (see, for example, Proposition 6.7 of [10]).

**Proposition 2.2.4.** *Let $X$ be a $\delta$-hyperbolic space. For all $K$, there exists $K' = K'(K, \delta)$ such that if $A$ is a set of at most $K$ points in $X$, then the following holds. There is a (piecewise geodesic) tree $\tau$ in $X$, containing $A$, with induced path metric $d_\tau$ on $\tau$, such that for all $a, b \in A$, we have $d_\tau(a, b) \leqslant d_X(a, b) + K'$.* □

## 2.3 Geometry of groups

An important motivation for concepts in large scale geometry is the study of the geometry of groups. Let $G$ be a group with a finite generating set $\mathcal{S}$. We can consider $G$ as a metric space as follows.

**Definition 2.3.1.** The *Cayley graph*, $\Delta(G, \mathcal{S})$, has a vertex for each element of $G$ and an edge joining $g$ and $h$ if $g^{-1}h \in \mathcal{S} \cup \mathcal{S}^{-1}$.

We give $\Delta(G,\mathcal{S})$ a metric by setting each edge to have length 1. Although this is dependent on the choice of generating set $\mathcal{S}$, we have the following (see, for example, Theorem 3.3 of [10]).

**Theorem 2.3.2.** *Let $\mathcal{S}$, $\mathcal{S}'$ be two finite generating sets for a group $G$. Then $\Delta(G,\mathcal{S})$ and $\Delta(G,\mathcal{S}')$ are quasi-isometric.* □

Hence large scale properties of $\Delta(G,\mathcal{S})$ can be considered as properties of $G$. For example, we have a notion of a *hyperbolic group.* We say that a group $G$ is quasi-isometric to a space $X$ if some Cayley graph for $G$ is quasi-isometric to $X$.

The group $G$ acts isometrically by left multiplication on any Cayley graph for $G$. More generally, we can consider isometric actions of a group $G$ on other metric spaces. A geodesic metric space $X$ is *proper* if every closed ball in $X$ is compact.

**Definition 2.3.3.** Suppose a group $G$ acts isometrically on a metric space $X$.

1. The action is *properly discontinuous* if, for all $x \in X$ and all $r \geqslant 0$, the set $\{g \in G \mid d_X(x, gx) \leqslant r\}$ is finite.

2. The action is *cocompact* if the quotient $X/G$ is compact.

The following is sometimes referred to as the Švarc–Milnor lemma (see Proposition I.8.19 of [17]).

**Theorem 2.3.4.** *Let a group $G$ act by isometries on a proper geodesic space $X$, and suppose that the action is properly discontinuous and cocompact. Then $G$ is quasi-isometric to $X$.* □

# Chapter 3

# Surfaces, curves and mapping class groups

In this chapter, we introduce mapping class groups, Teichmüller spaces, curve graphs and related graphs, and quote some results and methods which we shall use later. We will also describe a little of the history of the geometry of curve graphs (Section 3.5), and define the coarse median property (Section 3.6) and hierarchical hyperbolicity (Section 3.7).

## 3.1 Surfaces and curves

The surfaces we consider will be oriented, compact and connected, and hence homeomorphic to $S_{g,b}$ for some $g$ and $b$, where this notation refers to the genus $g$ surface with $b$ boundary components. We will abbreviate $S_{g,0} = S_g$. Note that we could alternatively allow a surface $S$ to have a finite number of punctures instead of (or as well as) boundary components. Replacing boundary components by punctures would affect various definitions in this thesis, but the results of Chapters 4 and 5 would go through unchanged. A reference for the definitions and results of this section and Section 3.2 is [26].

A *simple closed curve* in a surface $S$ is an embedding $\alpha\colon S^1 \hookrightarrow S$, or its image $\alpha(S^1)$, which we shall also denote by $\alpha$. A curve is *essential* if it does not bound a disc in the surface $S$ and *non-peripheral* if it does not cobound an annulus with a component of the boundary. From now on, any curve will be an essential, non-peripheral simple closed curve unless otherwise stated. A curve $\alpha$ is *separating* if $S \setminus \alpha$ is disconnected, and *non-separating* otherwise.

Recall that an embedding $f\colon X \hookrightarrow Y$ is *proper* if $f(\partial X) = f(X) \cap \partial Y$. An *arc* in $S$ is a proper embedding $a\colon [0,1] \hookrightarrow S$, or its image $a([0,1])$ in $S$. An arc $a$ is *essential* if no component of $S \setminus a$ is a disc whose boundary is the union of $a$ and a subarc of the boundary of $S$.

Recall that an *isotopy* between two curves $\alpha$ and $\beta$ is a homotopy between the maps $\alpha\colon S^1 \hookrightarrow S$ and $\beta\colon S^1 \hookrightarrow S$ where every intermediate map in the homotopy is also an embedding. In [24], Epstein proved that two curves are isotopic if and only if they are homotopic, a result due to Baer in the case of closed surfaces. For arcs, we will require an isotopy to be *proper*, that is, for every intermediate map, the endpoints of the arc are in $\partial S$.

We will typically consider curves only up to isotopy (although sometimes, in particular in Chapter 5, it will be convenient to work with fixed representatives of isotopy classes). Abusing notation, we will usually use $\alpha$ to denote the isotopy class of $\alpha$ as well as a specific representative.

The *intersection number* $i(\alpha, \beta)$ of two isotopy classes of curves $\alpha$ and $\beta$ is the minimal number of intersections between representative curves from the respective isotopy classes. Two curves $\alpha$ and $\beta$ are said to be in *minimal position* if they intersect transversely and the number of intersections between $\alpha$ and $\beta$ is $i(\alpha, \beta)$.

A *bigon* between $\alpha$ and $\beta$ is a disc in $S$ whose boundary is made up of an arc $a$ of $\alpha$ and an arc $b$ of $\beta$ intersecting only at their endpoints. Moreover, any arcs of intersection of $\alpha$ or $\beta$ with the interior of the bigon do not meet the points of $a \cap b$, that is, the two corners point "outwards". We have the following useful characterisation of minimal position (see, for example, Proposition 1.7 of [26]).

**Proposition 3.1.1.** *Let $\alpha$ and $\beta$ be curves in a surface $S$, intersecting transversely. Then $\alpha$ and $\beta$ are in minimal position if and only if they do not form a bigon.* $\square$

Whenever $S$ has negative Euler characteristic, we can equip $S$ with a hyperbolic metric. It is useful to observe that for any pair of curves in $S$ and any hyperbolic metric on $S$, the (unique) geodesic representatives of the two curves intersect minimally. Hence we may realise all curves in $S$ simultaneously in minimal position by fixing a hyperbolic metric on $S$ and taking the geodesic representative of each curve. For $S_1$, we can similarly fix a Euclidean metric on $S$ and choose geodesic representatives of curves.

We say that a collection of curves $A$ in $S$ *fills* $S$ if every other curve in $S$ has non-trivial intersection with some curve of $A$. Equivalently, $S \setminus A$ is a collection of topological discs and peripheral annuli.

A *multicurve* in $S$ is a set of pairwise disjoint, pairwise non-isotopic curves in $S$. Once again, we will typically consider multicurves up to isotopy. The definitions of intersection number and minimal position for multicurves are analogous to those for curves.

A multicurve in a surface $S$ will have a maximal number of curves precisely when its complement in $S$ is a collection of copies of $S_{0,3}$ ("pairs of pants"). Such a multicurve is called a *pants decomposition* of $S$. The *complexity*, $\xi(S)$, of $S$ is the number of curves in a pants decomposition of $S$. When $S = S_{g,b}$, we have $\xi(S) = 3g - 3 + b$.

## 3.2 The mapping class group and Teichmüller space

### 3.2.1 Mapping class groups

The *mapping class group* $\mathrm{MCG}(S)$ of a surface $S$ is the group of isotopy classes of orientation-preserving homeomorphisms of $S$ which fix the boundary pointwise. An element of this group is called a *mapping class*.

A *Dehn twist* about a curve $\alpha$ is defined by identifying a regular annular neighbourhood $N$ of $\alpha$ with an annulus $S^1 \times [0,1]$ and applying the twist map $(x,t) \mapsto (xe^{2\pi it}, t)$ (recall that we identify $S^1$ with the unit circle in $\mathbb{C}$). The isotopy class of this homeomorphism is determined by the isotopy class of the curve $\alpha$. Moreover, a Dehn twist about an essential curve has infinite order in $\mathrm{MCG}(S)$. See Chapter 3 of [26] for background on Dehn twists. The mapping class group is generated by a finite number of Dehn twists about curves in $S$ and components of $\partial S$. For closed surfaces, this is the Dehn–Lickorish Theorem. See Chapter 4 of [26] for a proof, including a discussion of the non-closed case.

If we choose to consider surfaces with a finite number of punctures instead of boundary components, then the mapping class group is slightly different as it may now permute punctures. Moreover any twist about a puncture is trivial in the mapping class group, whereas a twist about a boundary component is non-trivial.

Note that it is consistent to think about mapping classes along with isotopy classes of curves. Specifically, if $\phi$ and $\psi$ are two isotopic self-homeomorphisms of $S$ and $\alpha$ and $\beta$ are two isotopic curves in $S$ then $\phi(\alpha)$ is isotopic to $\psi(\beta)$. In fact, the mapping class group has an action on the set of isotopy classes of curves in $S$, and we shall return to this in Section 3.3.

One very useful fact about the action of the mapping class group on the curves in a surface is described in [26] (Section 1.3) as the *change of coordinates principle*. As an example, for any two non-separating curves $\alpha$ and $\beta$ in $S$, there is a mapping class taking $\alpha$ to $\beta$. The idea of proving statements of this kind is to apply the classification of surfaces to the surfaces formed by cutting along $\alpha$ or

$\beta$ to see that they are homeomorphic. Two separating curves $\alpha$ and $\beta$ in $S$ will be related by a mapping class if the components of $S \setminus \alpha$ are homeomorphic to the components of $S \setminus \beta$, preserving boundary components of $S$. More generally, we can apply this to multicurves, where the homeomorphisms should respect which boundary components of the complement of the multicurve in $S$ are identified by gluing along the multicurve. Even more generally, we can say that two sets of curves in $S$ with "the same intersection pattern" are related by a mapping class. An important consequence of the change of coordinates principle is that up to the action of the mapping class group there are only finitely many (multi)curves on $S$, and, for any $N$, only finitely many pairs of curves intersecting at most $N$ times.

Another important result is the Alexander method (see Proposition 2.8 of [26]). This states that if $A$ is a collection of curves and arcs which cut $S$ into topological discs, and $f$ is a mapping class which fixes the isotopy class of every curve and arc in $A$ , then $f$ is the identity. Hence any two mapping classes can be distinguished by their action on the set of curves and arcs in $S$ (indeed, on a finite subset of this set). When there are sufficiently many curves, arcs are needed only to detect twists about a boundary component.

### 3.2.2 Teichmüller space

The *Teichmüller space*, Teich($S$), of a surface $S$ can be thought of as parametrising hyperbolic structures on $S$. A *marked hyperbolic surface* $(X, f)$ is a complete, finite-volume, hyperbolic surface $X$ with totally geodesic boundary, together with a diffeomorphism $f \colon S \to X$. Two such marked hyperbolic surfaces $(X, f)$, $(Y, g)$ are equivalent if there exists an isometry $I \colon X \to Y$ such that $I \circ f$ is homotopic to $g$. A point of Teich($S$) is an equivalence class of marked hyperbolic surfaces. There is a natural topology on this set of points, and, in fact, Teich($S$) is homeomorphic to an open ball. There are a number of different metrics which have been defined for Teichmüller space, though we shall not give definitions here. Two metrics which have been studied extensively are the *Teichmüller metric* and the *Weil–Petersson metric*.

## 3.3 Complexes associated to surfaces

### 3.3.1 The curve graph

Central to the study of mapping class groups and Teichmüller spaces in recent years have been various simplicial complexes that can be associated to a surface, often equipped with a natural simplicial action of the mapping class group. The *curve complex* for a surface $S$ was introduced by Harvey in 1981 [32], and has a vertex for every isotopy class of essential, non-peripheral simple closed curves in $S$. A set of $k + 1$ distinct vertices spans a $k$-simplex if the corresponding isotopy classes have representatives on $S$ which are pairwise disjoint. This complex is a flag complex (that is, every complete graph on $n$ vertices in the 1-skeleton bounds an $(n-1)$-simplex), and so all combinatorial information is encoded in the 1-skeleton, the *curve graph*. Here, we will always consider the curve graph rather than the curve complex. We denote the curve graph by $\mathcal{C}(S)$, observing that this notation is also commonly used for the curve complex. The curve graph is equipped with the combinatorial metric $d_S$ given by setting each edge to have length 1. Since we shall only really be interested here in the distance between vertices, and not between other points in the graph, for notational convenience we will sometimes think of $\mathcal{C}(S)$ as a discrete set of vertices with the induced metric. A path in $\mathcal{C}(S)$ can then be thought of as a sequence of vertices where consecutive vertices in the sequence are at distance 1. Since $\mathrm{MCG}(S)$ acts on $\mathcal{C}(S)$ by simplicial automorphisms, it has an isometric action on $\mathcal{C}(S)$ with this metric. Note, however, that this action is not properly discontinuous, since the infinite cyclic subgroup of $\mathrm{MCG}(S)$ generated by the Dehn twist about a curve $\alpha$ fixes the vertex $\alpha$. Moreover, $\mathcal{C}(S)$ is not a proper metric space since each vertex has infinite degree.

Whenever $\xi(S) \geqslant 2$, the curve graph, $\mathcal{C}(S)$, is connected (see, for example, Lemma 2.1 of [41]). When $S$ is $S_{1,0}$, $S_{1,1}$ or $S_{0,4}$, we modify the definition so that two distinct curves are adjacent if they intersect minimally on $S$ (once for the first two cases and twice for the third). In each case, this modified graph is connected, and, in fact, the resulting graphs are isomorphic. This graph is the *Farey graph*. The curve graph of $S_{0,3}$ is empty since there are no essential, non-peripheral curves on $S_{0,3}$. However, we do define a curve graph for the annulus, $S_{0,2}$, which is more accurately a graph of arcs. We will not give a formal definition here as we shall not be using this graph for any of our results, but, loosely speaking, $\mathcal{C}(S_{0,2})$ records twisting about the core curve of the annulus.

### 3.3.2 Other graphs

There are many variations on the curve graph which give different information about the mapping class group and Teichmüller space. In particular, different graphs can tell us about different subgroups of $\mathrm{MCG}(S)$ (see, for example, Section 6 of [16]). We give just a few examples here.

The **separating curve graph**, $\mathrm{Sep}(S)$, is the full subgraph of $\mathcal{C}(S)$ which is spanned by separating curves. It is not quasi-isometrically embedded in $\mathcal{C}(S)$ (see Claim 2.41 of [54]). The separating curve graph has been applied by Brendle and Margalit to study properties of the Johnson kernel, a subgroup of the mapping class group [15]. Chapter 4 of this thesis concerns properties of the separating curve graph.

The **pants graph**, $\mathcal{P}(S)$, has a vertex for each pants decomposition of $S$, with edges corresponding to *elementary moves*. An elementary move involves choosing a curve $\alpha$ of the pants decomposition $P$, selecting the unique component $X_\alpha$ of $S \setminus (P \setminus \alpha)$ such that $\xi(X_\alpha) = 1$, and replacing $\alpha$ with a curve which is adjacent to $\alpha$ in $\mathcal{C}(X_\alpha)$. Brock proved that $\mathcal{P}(S)$ is quasi-isometric to the Teichmüller space of $S$ with the Weil–Petersson metric [18].

If $S$ is a boundary component of a compact, orientable 3-manifold $M$, then the **disc graph**, $\mathcal{D}(M, S)$, is the full subgraph of $\mathcal{C}(S)$ spanned by curves which bound embedded discs in $M$. Since the action of the mapping class group on the set of curves in $S$ does not preserve the property of bounding a disc in $M$, the graph $\mathcal{D}(M, S)$ does not have a natural action of $\mathrm{MCG}(S)$ as for the other examples above. However, it does have an action of the mapping class group of $M$. In particular, if $M$ is a handlebody, then the handlebody subgroup of $\mathrm{MCG}(S)$ acts on $\mathcal{D}(M, S)$. Section 5.2 of this thesis gives a new proof of the quasiconvexity of $\mathcal{D}(M, S)$ in $\mathcal{C}(S)$.

## 3.4 Subsurface projections

An *essential* subsurface of a surface $S$ is a connected subsurface $X$ so that every boundary component of $X$ is either a boundary component of $S$ or an essential, non-peripheral curve of $S$. From now on, the word "subsurface" will always refer to an isotopy class of essential subsurfaces. Note that the complexity $\xi(S)$ strictly decreases when taking proper subsurfaces. Given a subsurface $X$ of $S$, we define $\partial_S X$ to be $\partial X \setminus (\partial X \cap \partial S)$, that is, the multicurve of $S$ made up of the boundary components of $X$ which are not in $\partial S$.

Given a surface $S$ and a subsurface $X$ of $S$, we have a *subsurface projection* map $\pi_X$ from $\mathcal{C}(S)$ to the power set $2^{\mathcal{C}(X)}$ of $\mathcal{C}(X)$. As mentioned in Section 3.3.1, we here think of curve graphs and similar graphs as discrete sets of vertices. In particular, when we consider maps between curve graphs these will not necessarily be graph morphisms. The image of a vertex under the subsurface projection map may be empty, and always has uniformly bounded diameter (see Proposition 3.4.1 below). We define this subsurface projection for subsurfaces with positive complexity following [42]. A subsurface projection to the curve graph of an annulus can also be defined but we will not need it here. For a subsurface $X$ homeomorphic to $S_{0,3}$, we do not have a subsurface projection since the curve graph of $X$ is empty.

Now, let $X$ be a subsurface of $S$ with $\xi(S) \geqslant 1$, and let $\alpha$ be a curve of $S$ intersecting $X$ minimally. That is, $\alpha$ and $\partial_S X$ are in minimal position, and if $\alpha$ is isotopic to a boundary component of $X$ then it is isotoped to be disjoint from $X$. If $\alpha$ is contained in $X$ then $\pi_X(\alpha) = \alpha$. If $\alpha$ is disjoint from $X$ then $\pi_X(\alpha) = \varnothing$. Otherwise, the intersection of $\alpha$ and $X$ is a collection $\mathcal{A}$ of properly embedded arcs in $X$. Then $\pi_X(\alpha)$ is the set containing each essential, non-peripheral curve in $X$ which arises as a boundary component of a regular closed neighbourhood of the union of some $a$ in $\mathcal{A}$ and the components of $\partial_S X$ it meets (see Figure 3.1 for examples). We may similarly consider a subsurface projection $\mathcal{G}(S) \to \mathcal{C}(X)$ for any complex $\mathcal{G}(S)$ whose vertices are curves or multicurves in $S$, and any subsurface $X$ of $S$. If $B$ is a collection of curves, then $\pi_X(B) = \bigcup_{\alpha \in B} \pi_X(\alpha)$.

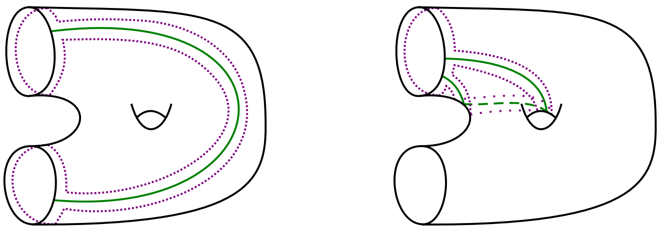


Figure 3.1: Examples of subsurface projection.

We define the distance between two sets $C$, $D$ of curves in $X$ by $d_X(C, D) = \mathrm{diam}_{\mathcal{C}(X)}(C \cup D)$. We usually abbreviate $d_X(\pi_X(A), \pi_X(B))$ by $d_X(A, B)$. The following result is included in Lemma 2.3 of [42].

**Proposition 3.4.1.** *Let $X$ be a subsurface of $S$ of positive complexity and let $a$ be a multicurve in $S$. Then either $\pi_X(a) = \varnothing$ or $\mathit{diam}_{\mathcal{C}(X)}(\pi_X(a)) \leqslant 2$.* □

This implies that if $\alpha_0, \alpha_1, \ldots, \alpha_n$ is a path in $\mathcal{C}(S)$ such that every $\alpha_i$ intersects $X$, then $d_X(\alpha_0, \alpha_n) \leqslant 2n$.

Given a complex $\mathcal{G}(S)$, the subsurfaces of $S$ which every vertex of $\mathcal{G}(S)$ must intersect are of particular interest. These are called *holes* in [45], and *witnesses* in some more recent papers (see, for example, [3, 22]).

## 3.5 Properties of curve complexes and applications

Harvey introduced the curve complex in [32] in order to study a bordification of Teich($S$) and the action of MCG($S$) on this space. Another early use of the curve complex was by Harer, to study homological properties of the mapping class groups (see, for example, [30, 31]). Again, the full complex (and not just the 1-skeleton) was used, and Harer proved that it is homotopy equivalent to a wedge of spheres, all of the same dimension. Another major contributor to early work on the curve complex was Ivanov. For example, in [36], Ivanov proved that the automorphism group of the curve complex is the extended mapping class group (given by allowing orientation-reversing homeomorphisms as well as orientation-preserving ones). He used this to give a new proof of a result of Royden [52] and of Earle and Kra [23] that every isometry of the Teichmüller metric is induced by an element of this group, as well as to investigate some algebraic properties of $\mathrm{MCG}(S)$.

In two papers [41, 42], Masur and Minsky linked the large scale geometry of the curve graph to the geometry of the mapping class group and Teichmüller space (note that the surfaces in these papers have punctures rather than boundary). In [41], they proved (Theorem 1.1) that for each surface $S$, there exists $\delta$ such that $\mathcal{C}(S)$ is $\delta$-hyperbolic, with infinite diameter whenever $\xi(S) \geqslant 1$. Moreover, they draw conclusions about the geometry of $\mathrm{Teich}(S)$ and $\mathrm{MCG}(S)$. Teichmüller space, with the Teichmüller metric, is not $\delta$-hyperbolic. Specifically, we can define regions $H_\alpha$ in $\mathrm{Teich}(S)$ which correspond to metrics where a curve $\alpha$ is short, and these regions look like products. As a consequence of the Collar Lemma (see [38]), two intersecting curves cannot both be short in the same hyperbolic metric on $S$, but two disjoint curves can both be short. Hence, $\mathcal{C}(S)$ encodes the intersections of the regions $H_\alpha$. We can say that $\mathcal{C}(S)$ is the *nerve* of this family of regions. We can cone off a region by adding a point at distance $\frac{1}{2}$ from every point in this region. Theorem 1.2 of [41] states that if the regions $H_\alpha$ are coned off then the space obtained is quasi-isometric to $\mathcal{C}(S)$, and hence $\delta$-hyperbolic. These regions can be thought of as the obstructions to the hyperbolicity of the Teichmüller metric. Theorem 1.3 of [41] gives a similar result for $\mathrm{MCG}(S)$. In this case, subgroups which fix curves in $S$ give products within $\mathrm{MCG}(S)$. Coning off certain such subgroups with their cosets again gives a space quasi-isometric to $\mathcal{C}(S)$.

In [42], Masur and Minsky use subsurface projections to study the geometry of $\mathrm{MCG}(S)$. They define a graph $\mathcal{M}(S)$ called the *marking graph* which is quasi-isometric to $\mathrm{MCG}(S)$ and define *hierarchies* of geodesics in curve graphs of subsurfaces of $S$ to study paths in this graph. Using this machinery, they prove that $\mathcal{M}(S)$ (and hence $\mathrm{MCG}(S)$) has a *distance formula* in terms of a sum of subsurface projections to all subsurfaces of $S$ (including annuli). More precisely, in Theorem 6.12 of [42] they show the following, where $[x]_C$ is equal to $x$ when $x \geqslant C$ and 0 otherwise.

**Theorem 3.5.1.** *There exists $C_0$ such that, for all $C \geqslant C_0$, there exist $K_1$ and $K_2$*

*such that, for any two markings $\mu$ and $\nu$ we have:*

$$d_{\mathcal{M}(S)}(\mu,\nu) \asymp_{K_1,K_2} \sum_{X\subseteq S} [d_X(\pi_X(\mu),\pi_X(\nu))]_C\,. \qquad \square$$

## 3.6 The coarse median property

In [11], Bowditch introduced the concept of a *coarse median space.* Mapping class groups of surfaces are motivating examples of such spaces, along with all $\delta$-hyperbolic spaces and CAT(0) cube complexes (see below for a definition).

The definition of a coarse median space uses the concept of a median algebra. See, for example, [4] for a survey.

**Definition 3.6.1.** A *median algebra* $(M,\mu)$ is a set $M$ with a ternary operation $\mu : M^3 \to M$ such that, for all $a,b,c,d,e \in M$:

(M1) $\mu(a,b,c) = \mu(b,c,a) = \mu(b,a,c)$,

(M2) $\mu(a,a,b) = a$,

(M3) $\mu(a,b,\mu(c,d,e)) = \mu(\mu(a,b,c),\mu(a,b,d),e)$.

A finite median algebra can equivalently be viewed as the vertex set of a finite CAT(0) cube complex. We give an overview below; see [51] for details. The term CAT(0) refers to a non-positive curvature condition for a metric space, which is defined in terms of measurements of triangles in the space. However, in the specific case of cube complexes there is a more combinatorial characterisation. We now give a brief definition of CAT(0) cube complexes; see, for example, [53] for more details.

We build a cube complex from unit Euclidean cubes $[0,1]^n$ for various $n$, glued by isometries between faces. Recall that a *flag complex* $K$ is a simplicial complex such that every complete graph with $n$ edges in the 1-skeleton of $K$ bounds an $(n-1)$-simplex in $K$. The following can be taken as a definition of a CAT(0) cube complex, though it also coincides with the metric definition of CAT(0) for cube complexes.

**Definition 3.6.2.** A connected cube complex $X$ is CAT(0) if it is simply-connected and the link of every vertex in $X$ is a flag complex.

We can define a median operation on the vertices of a CAT(0) cube complex $X$ in the following way. For two vertices $x$, $y$ of $X$, define $[x,y]$ to be the set of all vertices of $X$ which lie in some geodesic between $x$ and $y$ in the 1-skeleton $X^{(1)}$ of $X$. Given three vertices $x$, $y$ and $z$, the sets $[x,y]$, $[x,z]$ and $[y,z]$ intersect in a unique point which we call $\mu(x, y, z)$. This point is the closest point projection of $x$ to $[y, z]$ in $X^{(1)}$. One can check that the ternary operation $\mu$ on the vertices of $X$ satisfies the conditions of Definition 3.6.1. Specifically, (M1) states that it does not matter in which order we take the points, (M2) holds because $[a, a]$ is simply $a$ itself, and (M3) can be interpreted as saying that certain projections commute. Hence, this gives the vertex set of $X$ the structure of a median algebra. In fact, every finite median algebra can be canonically identified as the vertex set of a finite CAT(0) cube complex (Theorem 10.3 of [51]). We can take the following to be a definition of the rank of a median algebra.

**Definition 3.6.3.** Let $\Pi$ be a finite median algebra and $X$ the CAT(0) cube complex identified with $\Pi$. The *rank* of $\Pi$ is the dimension of $X$.

A coarse median space is equipped with a ternary operation called a *coarse median* which approximates to the median operation on a finite median algebra for any finite set of points in the space. In particular, any triple of points in the space has a coarsely well defined centre. The two following motivating examples are described in [11]. For a hyperbolic space the coarse median of three points can be defined to be a centre for a geodesic triangle (see Section 2.2). For the mapping class group of a surface, the coarse median operation can be taken to be the *centroid* defined by Behrstock and Minsky in [9].

**Definition 3.6.4.** A ternary operation $\mu : \Lambda^3 \to \Lambda$ on a geodesic space $(\Lambda, d)$ is a *coarse median* if:

(C1) there exist $k$, $h$ such that for all $a$, $b$, $c$, $a'$, $b'$, $c'$ in $\Lambda$,

$$d(\mu(a, b, c), \mu(a', b', c')) \leqslant k(d(a, a') + d(b, b') + d(c, c')) + h,$$

(C2) for every $p \in \mathbb{N}$ there exists $q$ such that if $A$ is a subset of $\Lambda$ with at most $p$ elements then there exist a finite median algebra $(\Pi, \mu_\Pi)$ and maps $\pi : A \to \Pi$, $\lambda : \Pi \to \Lambda$ such that for all $x$, $y$, $z$ in $\Pi$,

$$d(\lambda(\mu_\Pi(x, y, z)), \mu(\lambda(x), \lambda(y), \lambda(z))) \leqslant q,$$

and for all $a$ in $A$, we have $d(a, \lambda(\pi(a))) \leqslant q$.

The coarse median property is a quasi-isometry invariant. A useful property of a coarse median space is its associated rank, which is also invariant under quasi-isometries.

**Definition 3.6.5.** A coarse median space $\Lambda$ has *rank* $\nu$ if for any finite set $A$ of points in $X$, the median algebra $\Pi$ as in Definition 3.6.4 can be chosen to have rank at most $\nu$, and if this is not possible for $\nu - 1$.

We will quote some results on properties of coarse median spaces in Section 3.7.2.

## 3.7 Hierarchical hyperbolicity

### 3.7.1 Definition

Hierarchically hyperbolic spaces were defined by Behrstock, Hagen and Sisto in [6]. The same authors give an equivalent definition of hierarchically hyperbolic spaces in [7], and that is the definition we shall state below. For an exposition of the topic of hierarchically hyperbolic spaces, see [55]. Every hierarchically hyperbolic space is also a coarse median space [13, 7] (see Theorem 3.7.4 below). Mapping class groups of surfaces are motivating examples, and the construction is inspired by the work of Masur and Minsky in [42]. Hierarchical hyperbolicity of a space $\Lambda$ is always with respect to some family of uniformly hyperbolic spaces with projections from $\Lambda$ to these spaces. The space $\Lambda$ is assumed to be a *quasigeodesic space*, that is, any two points in the space can be connected by a quasigeodesic with uniform constants.

We say that $(\Lambda, d_\Lambda)$ is a *hierarchically hyperbolic space* if there exist a constant $\delta \geqslant 0$, an indexing set $\mathfrak{S}$ and, for each $X \in \mathfrak{S}$, a $\delta$-hyperbolic space $(\mathcal{C}(X), d_X)$ such that the following axioms are satisfied (see Definition 1.1 of [7]).

**1. Projections.** There exist constants $c$ and $K$ such that for each $X \in \mathfrak{S}$, there is a $(K, K)$-coarsely Lipschitz *projection* $\pi_X : \Lambda \to 2^{\mathcal{C}(X)}$ such that the image of each point of $\Lambda$ has diameter at most $c$ in $\mathcal{C}(X)$.

**2. Nesting.** The set $\mathfrak{S}$ has a partial order $\sqsubseteq$, and if $\mathfrak{S}$ is non-empty then it contains a unique $\sqsubseteq$-maximal element. If $X \sqsubseteq Y$ then we say that $X$ is *nested* in $Y$. For all $X \in \mathfrak{S}$, we have $X \sqsubseteq X$. For all $X, Y \in \mathfrak{S}$ such that $X \sqsubsetneq Y$ (that is, $X \sqsubseteq Y$ and $X \neq Y$), there is an associated subset $\pi_Y(X) \subseteq \mathcal{C}(Y)$ with diameter at most $c$, and a projection map $\pi^Y_X : \mathcal{C}(Y) \to 2^{\mathcal{C}(X)}$.

**3. Orthogonality.** There is a symmetric and anti-reflexive relation $\perp$ on $\mathfrak{S}$ called *orthogonality*, satisfying the following.

- Whenever $Y \sqsubseteq X$ and $X \perp Z$, we have $Y \perp Z$.
- For every $X \in \mathfrak{S}$ and $Y \sqsubseteq X$, either there is no $U \sqsubseteq X$ such that $U \perp Y$, or there exists $Z \sqsubsetneq X$ such that whenever $U \sqsubseteq X$ and $U \perp Y$, we have $U \sqsubseteq Z$.

- If $X \perp Y$ then $X$ and $Y$ are not $\sqsubseteq$-comparable, that is, neither is nested in the other.

**4. Transversality and consistency.** If $X$ and $Y$ are not orthogonal and neither is nested in the other, then we say $X$ and $Y$ are *transverse*, $X \pitchfork Y$. There exists $\kappa \geqslant 0$ such that whenever $X \pitchfork Y$ there are sets $\pi_X(Y) \subseteq \mathcal{C}(X)$ and $\pi_Y(X) \subseteq \mathcal{C}(Y)$, each of diameter at most $c$, satisfying, for all $a \in \Lambda$:

$$\min\{d_X(\pi_X(a), \pi_X(Y)), d_Y(\pi_Y(a), \pi_Y(X))\} \leqslant \kappa.$$

If $X \sqsubseteq Y$ and $a \in \Lambda$ then:

$$\min\{d_Y(\pi_Y(a), \pi_Y(X)), \operatorname{diam}_{\mathcal{C}(X)}(\pi_X(a) \cup \pi_X^Y(\pi_Y(a)))\} \leqslant \kappa.$$

These are called the *consistency inequalities*.

Moreover, if $Y \sqsubseteq X$, and if $Z$ is such that each of $X$ and $Y$ is either strictly nested in $Z$ or transverse to $Z$, then $d_Z(\pi_Z(X), \pi_Z(Y)) \leqslant \kappa$.

**5. Finite complexity.** There exists $n \geqslant 0$, called the *complexity* of $\Lambda$ with respect to $\mathfrak{S}$, such that any set of pairwise $\sqsubseteq$-comparable elements of $\mathfrak{S}$ contains at most $n$ elements.

**6. Large links.** There exist $\lambda \geqslant 1$ and $E \geqslant \max\{c, \kappa\}$ such that the following holds. Let $X \in \mathfrak{S}$, $a, b \in \Lambda$ and $R = \lambda d_X(\pi_X(a), \pi_X(b)) + \lambda$. Then either $d_Y(\pi_Y(a), \pi_Y(b)) \leqslant E$ for every $Y \sqsubsetneq X$, or there exist $Y_1, \ldots, Y_{\lfloor R \rfloor}$ in $\mathfrak{S}$ such that for each $1 \leqslant i \leqslant \lfloor R \rfloor$, $Y_i \sqsubsetneq X$, and such that for all $Y \sqsubsetneq X$, either $Y \sqsubseteq Y_i$ for some $i$, or $d_Y(\pi_Y(a), \pi_Y(b)) \leqslant E$. Moreover, $d_X(\pi_X(a), \pi_X(Y_i)) \leqslant R$ for each $i$.

**7. Bounded geodesic image.** For all $X, Y \in \mathfrak{S}$, with $Y \sqsubsetneq X$, and for all geodesics $g$ of $\mathcal{C}(X)$, either $\operatorname{diam}_{\mathcal{C}(Y)}(\pi_Y^X(g)) \leqslant E$ or $g \cap N_{\mathcal{C}(X)}(\pi_X(Y), E) \neq \varnothing$.

**8. Partial realisation.** There exists a constant $r$ with the following property. Let $\{X_j\}$ be a set of pairwise orthogonal elements of $\mathfrak{S}$ and let $\gamma_j \in \pi_{X_j}(\Lambda) \subseteq \mathcal{C}(X_j)$ for each $j$. Then there exists $a \in \Lambda$ such that:

- $d_{X_j}(\pi_{X_j}(a), \gamma_j) \leqslant r$ for all $j$,
- for each $j$ and each $X \in \mathfrak{S}$ such that $X_j \sqsubseteq X$, $d_X(\pi_X(a), \pi_X(X_j)) \leqslant r$,
- if $Y \pitchfork X_j$ for some $j$, then $d_Y(\pi_Y(a), \pi_Y(X_j)) \leqslant r$.

**9. Uniqueness.** For all $K \geqslant 0$, there exists $K'$ such that if $a, b \in \Lambda$ satisfy $d_X(\pi_X(a), \pi_X(b)) \leqslant K$ for all $X \in \mathfrak{S}$, then $d_\Lambda(a, b) \leqslant K'$.

### 3.7.2 Properties

An important basic property is that hierarchical hyperbolicity is a quasi-isometry invariant (Proposition 1.7 of [7]). This can be verified by composing the projections with the quasi-isometry.

**Proposition 3.7.1.** *If $\Lambda$ is hierarchically hyperbolic with respect to $\mathfrak{S}$ and $\Lambda'$ is a quasigeodesic space quasi-isometric to $\Lambda$, then $\Lambda'$ is hierarchically hyperbolic with respect to $\mathfrak{S}$.* □

It is shown in [7] (Theorem 5.5) that hierarchically hyperbolic spaces satisfy the following distance estimate, which generalises the result for mapping class groups given by Masur and Minsky in [42] (see Theorem 3.5.1 here). This was one of the axioms for the original definition of hierarchical hyperbolicity in [6] but is a consequence of the modified axioms in [7].

**Theorem 3.7.2.** *Let $\Lambda$ be hierarchically hyperbolic with respect to a set $\mathfrak{S}$. Then there exists a constant $C_0$ such that for all $C \geqslant C_0$ there exist $K_1$ and $K_2$ such that the following holds. For every $a, b \in \Lambda$, we have:*

$$d_\Lambda(a,b) \asymp_{K_1,K_2} \sum_{X \in \mathfrak{S}} \left[d_X(\pi_X(a), \pi_X(b))\right]_C . \qquad \square$$

Theorem J of [6] gives an upper bound on the dimension of a Euclidean space which can be quasi-isometrically embedded in a hierarchically hyperbolic space $\Lambda$ in terms of the maximal cardinality of a set of pairwise orthogonal elements of $\mathfrak{S}$. We obtain a stronger result by combining the following two results.

**Theorem 3.7.3.** *Let $\Lambda$ be a coarse median space of rank $d$, and fix some quasi-isometry constants. Then there exists $r$, depending only on $\Lambda$ and the quasi-isometry constants, such that there is no quasi-isometric embedding of the $(d+1)$-dimensional Euclidean ball of radius $r$ into $\Lambda$.* □

**Theorem 3.7.4.** *Let $\Lambda$ be hierarchically hyperbolic with respect to $\mathfrak{S}$ and let $d$ be the maximal cardinality of a set of pairwise orthogonal elements of $\mathfrak{S}$. Then $\Lambda$ is a coarse median space of rank at most $d$.* □

Theorem 3.7.3 is Lemma 6.10 of [13]. Theorem 3.7.4 is observed in [13], without the specific bound on rank. A proof, again without this bound on rank, is given in [7] (Theorem 7.3). However, one may verify that under the assumptions of Theorem 3.7.4, properties (P1)–(P4) of Section 10 of [11] are satisfied, with $\nu = d$, and hence, by Proposition 10.2 of that paper, $\Lambda$ is coarse median of rank at most $d$.

Another result on rank for coarse median spaces is the following, Theorem 2.1 of [11] (see also Corollary 4.3 of [47]).

**Theorem 3.7.5.** *Let $\Lambda$ be a coarse median space of rank 1. Then $\Lambda$ is Gromov hyperbolic.* $\square$

Any coarse median space satisfies a quadratic isoperimetric inequality, in the sense we shall describe below (Proposition 8.2 of [11]). See, for example, Section III.H.2 of [17] for more background on isoperimetric inequalities.

**Definition 3.7.6.** Let $\Lambda$ be a metric space, and $l, L > 0$.

- An *$l$-cycle* in $\Lambda$ of length $p$ is a set of points $a_0, a_1, \ldots, a_p = a_0$ in $\Lambda$ such that $d_\Lambda(a_i, a_{i+1}) \leqslant l$ for all $i$.
- An *$L$-disc* is a triangulation $T$ of the disc $D^2$, with a map $b\colon T^{(0)} \to \Lambda$ from its vertex set to $\Lambda$, such that if $x$ and $y$ in $T^{(0)}$ are connected by an edge of $T$, then $d_\Lambda(b(x), b(y)) \leqslant L$.
- An $l$-cycle $(a_i)_i$ *bounds* an $L$-disc $(T, b)$ if the vertices in $T^{(0)} \cap \partial D^2$ can be labelled by $x_i$ so that $x_i$ and $x_{i+1}$ are joined by an edge for all $i$, and so that $a_i = b(x_i)$ for all $i$.

**Theorem 3.7.7.** *Let $\Lambda$ be a coarse median space. For any $l > 0$ there exists $L > 0$ such that the following holds. For any $p \in \mathbb{N}$, any $l$-cycle in $\Lambda$ of length at most $p$ bounds an $L$-disc with at most $p^2$ 2-simplices in the triangulation.* $\square$

# Chapter 4

# Hierarchical hyperbolicity of the separating curve graph

In this chapter, we prove that the separating curve graph associated to a surface $S$ is a hierarchically hyperbolic space whenever it is connected. For background on hierarchically hyperbolic spaces, see Section 3.7. The work of this section appears in [58].

## 4.1 Preliminaries

### 4.1.1 Statement of results

The *separating curve graph*, $\mathrm{Sep}(S)$, of a surface $S$ is the full subgraph of $\mathcal{C}(S)$ spanned by all separating curves, with the combinatorial metric. Unlike the curve graph, the separating curve graph of a surface is not in general Gromov hyperbolic. We shall show that it is, however, a hierarchically hyperbolic space. The specific result we shall prove is the following theorem and immediate corollary (see Theorem 3.7.2), where $\mathfrak{X}$ is the set of subsurfaces $X$ of $S$ such that every separating curve intersects $X$ non-trivially. The excluded cases are those for which $\mathrm{Sep}(S)$ is not connected with the usual definition. For completeness, we give a proof of connectedness of $\mathrm{Sep}(S)$, for $S$ as in Theorem 4.1.1, in Section 4.1.3.

**Theorem 4.1.1.** *Let $S$ be a connected, compact, orientable surface. Suppose $S$ is not $S_{2,b}$ for $b \leqslant 1$, $S_{1,b}$ for $b \leqslant 2$ or $S_{0,b}$ for $b \leqslant 4$. Then the separating curve graph of $S$ is a hierarchically hyperbolic space with respect to subsurface projections to the curve graphs of subsurfaces in $\mathfrak{X}$.*

**Corollary 4.1.2.** *Let $S$ be as in Theorem 4.1.1. Then there exists a constant $C_0$ such that for every $C \geqslant C_0$ there exist $K_1$ and $K_2$ such that the following holds. For every pair of separating curves $\alpha, \beta$, we have:*

$$d_{Sep(S)}(\alpha, \beta) \asymp_{K_1, K_2} \sum_{X \in \mathfrak{X}} [d_X(\alpha, \beta)]_C . \qquad \square$$

The key step in proving Theorem 4.1.1 is to show that if the distance between the subsurface projections of two separating curves to $\mathcal{C}(X)$ is bounded by some $K$ for all $X \in \mathfrak{X}$, then there is a bound on their distance in $\mathrm{Sep}(S)$ depending only on $K$ and $\xi(S)$. We shall in fact verify this, along with the other conditions for hierarchical hyperbolicity, for a different graph, $\mathcal{K}(S)$, in Section 4.2. We will then show that $\mathcal{K}(S)$ is quasi-isometric to $\mathrm{Sep}(S)$ (Proposition 4.3.1). We remark that a complex similar to $\mathcal{K}(S)$ (the "complex of separating multicurves") is introduced by Sultan in [56], though, unlike $\mathcal{K}(S)$ and $\mathrm{Sep}(S)$, this complex is Gromov hyperbolic for every surface of sufficient complexity (Remark 3.1.9 of [56]). Sultan uses this complex to study the Weil–Petersson metric on Teichmüller space.

Using results quoted in Section 3.7.2, we obtain the corollaries below.

**Corollary 4.1.3.** *Let $S$ be as in Theorem 4.1.1. Then Sep($S$) satisfies a quadratic isoperimetric inequality in the sense of Theorem 3.7.7.* $\square$

**Corollary 4.1.4.** *Let $S = S_{g,b}$ be as in Theorem 4.1.1. Then there is no quasi-isometric embedding of the $n$-dimensional Euclidean space or half-space into Sep($S$), where $n = 3$ if $b \leqslant 2$ and $n = 2$ otherwise. In fact, for the same $n$, the radius of an $n$-dimensional Euclidean ball which can be quasi-isometrically embedded into Sep($S$) is bounded above in terms of $\xi(S)$ and the quasi-isometry constants.* $\square$

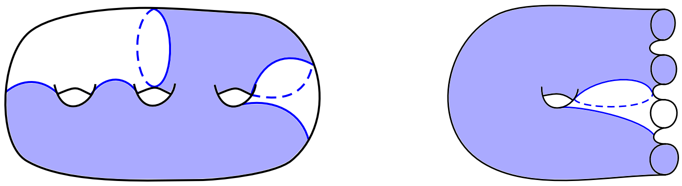
In other words, when $b \leqslant 2$, $\mathrm{Sep}(S)$ can have quasiflats of dimension 2 but not of any higher dimension. Such quasiflats correspond to pairs of disjoint subsurfaces in $\mathfrak{X}$; see Section 4.1.2 for a description of these. When $b > 2$, $\mathrm{Sep}(S)$ has no quasiflats of any dimension greater than 1. More detail on how quasiflats can behave in a hierarchically hyperbolic space is given by Behrstock, Hagen and Sisto in [8]. The fact that when $b > 2$ there are no pairs of disjoint subsurfaces in $\mathfrak{X}$ moreover implies the following.

**Corollary 4.1.5.** *Let $S = S_{g,b}$ be as in Theorem 4.1.1, with $b > 2$. Then Sep($S$) is Gromov hyperbolic.* $\square$

### 4.1.2 Subsurfaces in $\mathfrak{X}$

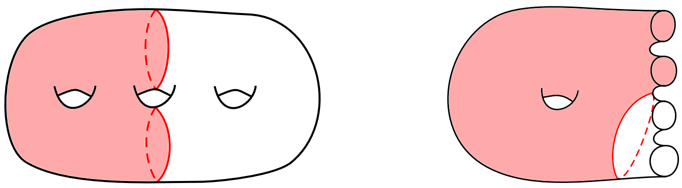
Recall that we defined $\mathfrak{X}$ to be the set of subsurfaces of $S$ which every separating curve intersects non-trivially. We will show that $\mathrm{Sep}(S)$ has a hierarchically hyperbolic structure with respect to $\mathfrak{X}$, where the associated hyperbolic spaces are the curve graphs of the subsurfaces in $\mathfrak{X}$. We briefly describe here what the subsurfaces in $\mathfrak{X}$ look like. To obtain compact surfaces, when we take the complement of a subsurface $X$ in $S$, we will then take the closure of this. However, for brevity, we will write simply $S \setminus X$. Similarly, when we remove a multicurve $a$ we will really want to remove a regular open neighbourhood, but again we will simply write $S \setminus a$.

Let $X \in \mathfrak{X}$. Then every component of $\partial_S X$ is non-separating in $S$ and no component of $S \setminus X$ contains a separating curve of $S$. Hence, each component of $S \setminus X$ is a planar subsurface containing at most one boundary component of $S$. Conversely, if $X$ is a subsurface such that every component of $S \setminus X$ is planar and contains at most one component of $\partial S$, then $X$ is in $\mathfrak{X}$. See Figure 4.1 for examples and Figure 4.2 for non-examples.



(a) (b)

Figure 4.1: Examples of subsurfaces which every separating curve must intersect.



(a) (b)

Figure 4.2: Examples of subsurfaces where there is a disjoint separating curve.

The relation of orthogonality for elements of $\mathfrak{X}$ will correspond to disjointness, so to obtain Corollary 4.1.4 from Theorem 3.7.3 and Theorem 3.7.4, we need to consider when a collection of subsurfaces in $\mathfrak{X}$ can be pairwise disjoint. First suppose that $S$ has at least three boundary components. Suppose that $X$ and $Y$ are disjoint subsurfaces, both contained in $\mathfrak{X}$. Since $Y$ is in $\mathfrak{X}$, so is any subsurface containing $Y$, so we can assume $Y$ is a component of $S \setminus X$. From the above discussion, every component of $S \setminus X$ is planar and contains at most one boundary component of $S$. Now, $S \setminus Y$ is connected and contains at least two boundary components of $S$, since $Y$ contains at most one. However, this contradicts that $Y$ is in $\mathfrak{X}$. Hence, if $S$ has at least three boundary components, then the maximal cardinality of a set of pairwise disjoint subsurfaces in $\mathfrak{X}$ is 1.

Now suppose that $S$ has at most two boundary components, and suppose that $X$ and $Y$ are disjoint subsurfaces in $\mathfrak{X}$. Again we can assume that $Y$ is a component of $S \setminus X$. Since $Y \in \mathfrak{X}$, the subsurface $S \setminus Y$ is planar and contains at most one component of $\partial S$. Suppose first that $S \setminus X$ is disconnected, and let $Z$ be a component of $S \setminus X$ other than $Y$. If $Z$ meets $X$ in more than one curve then $S \setminus Y$ has genus, which contradicts $Y \in \mathfrak{X}$. However, since $Z$ must be planar and must contain at most one component of $\partial S$, we have that $Z$ must be a disc or a peripheral annulus, contradicting that $X$ is an essential subsurface. Hence, $S \setminus X = Y$. Each of $X$ and $Y$ must be planar and must contain at most one component of $\partial S$. Now, suppose that $Y$ can be divided into two disjoint subsurfaces $V$ and $W$ in $\mathfrak{X}$. From above, the complement of each of these in $S$ must be connected. Moreover, since each of $X$, $V$ and $W$ has planar complement in $S$, we have that each of these subsurfaces is planar and that they pairwise meet in a single curve. Since $S$ has at most two boundary components, one of $X$, $V$ and $W$ must contain no component of $\partial S$. However, then the only possibility is that this subsurface is an annulus, which cannot be in $\mathfrak{X}$ for the surfaces we are considering.

Hence, if $S$ has at most two boundary components then a set of pairwise disjoint elements of $\mathfrak{X}$ can have cardinality 2, but not 3. Moreover, a pair of disjoint subsurfaces $X_1$, $X_2$ in $\mathfrak{X}$ must be arranged as follows (see Figure 4.3 for pictures for $g = 3$). If $S = S_g$, then each of $X_1$, $X_2$ is a copy of $S_{0,g+1}$, and they meet along all their boundary components (Figure 4.3a). If $S = S_{g,1}$, either $X_1$ and $X_2$ are both copies of $S_{0,g+1}$ (Figure 4.3b) or one is $S_{0,g+1}$ and one is $S_{0,g+2}$ (Figure 4.3c), and if $S = S_{g,2}$, then $X_1$ and $X_2$ are both copies of $S_{0,g+2}$ (Figure 4.3d). Notice that in most cases, if $X_1$ is a subsurface in $\mathfrak{X}$ such that there exists $X_2 \in \mathfrak{X}$ disjoint from $X_1$, then $X_2$ must be equal to $S \setminus X_1$ and hence is completely determined by $X_1$. The exception is when $S = S_{g,1}$ and $X_1$ is a copy of $S_{0,g+1}$. Then we may choose a curve $\gamma$ in $Y = S \setminus X_1$ such that one component of $Y \setminus \gamma$ is a copy of $S_{0,3}$ containing $\partial S$ and the other component is in $\mathfrak{X}$.

### 4.1.3 Connectedness of the separating curve graph

Here we give a proof of the connectedness of $\mathrm{Sep}(S)$ when $S = S_{g,b}$ is not $S_{0,b}$, $b \leqslant 4$, $S_{1,b}$, $b \leqslant 2$ or $S_{2,b}$, $b \leqslant 1$. This is a well known result (see Exercise 2.44 of [54]) but

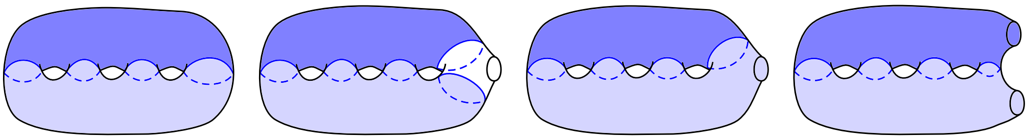



Figure 4.3: The possibilities for pairs of disjoint subsurfaces in $\mathfrak{X}$, up to $\mathrm{MCG}(S)$, for $S_3$, $S_{3,1}$ and $S_{3,2}$.

we have been unable to find a proof in the literature which covers all cases. In the case that $S$ is a closed surface of genus at least 3, the result appears in [25] and [39]; see also [44] and [49]. When $S$ has genus 0, every curve is separating, so $\mathrm{Sep}(S)$ is the usual curve graph (when $S$ has at least five boundary components). See, for example, [41] for a proof of connectedness of $\mathcal{C}(S)$ whenever it holds. Furthermore, stronger connectivity results which imply connectedness of $\mathrm{Sep}(S)$ when the genus of $S$ is at least 2, and $S$ is not $S_{2,0}$ or $S_{2,1}$, are given in [37].

We shall use the well known fact that a simple closed curve in $S$ is separating (including possibly inessential or peripheral) if and only if it is trivial in $H_1(S, \partial S; \mathbb{Z})$.

Let $\alpha$ and $\beta$ be two (essential, non-peripheral) separating curves in $S$. We shall assume for induction that for any separating curve $\gamma$ such that $i(\gamma, \beta) < i(\alpha, \beta)$, there is a path in $\mathrm{Sep}(S)$ from $\gamma$ to $\beta$. The base case is when $i(\alpha, \beta) = 0$, in which case $\alpha$ and $\beta$ are connected by an edge.

Now suppose $i(\alpha, \beta) \geqslant 2$ (the intersection number must always be even since the curves are separating). Assume that $\alpha$ and $\beta$ are in minimal position, so there is no bigon between $\alpha$ and $\beta$. Suppose first that one of the components $Y$ of $S \setminus \alpha$ either has genus at least 2, or has genus 1 and contains at least two boundary components of $S$, or is planar and contains at least three boundary components of $S$. We shall find a separating curve $\gamma$ such that $\gamma$ is disjoint from $\alpha$ (so adjacent to $\alpha$ in $\mathrm{Sep}(S)$) and such that $i(\gamma, \beta) < i(\alpha, \beta)$. Then $\gamma$ is connected to $\beta$ by the induction hypothesis, and so there is a path in $\mathrm{Sep}(S)$ from $\alpha$ to $\beta$.

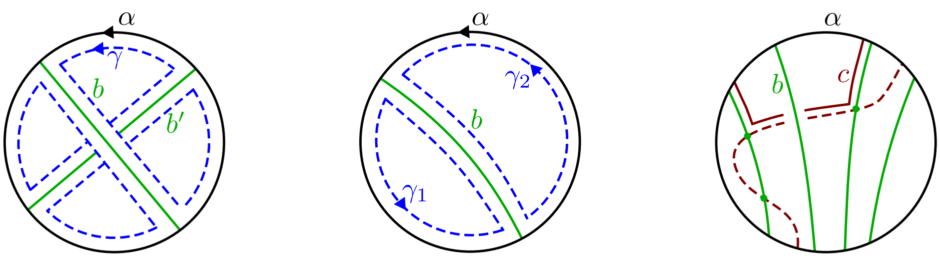
**Case 1.** Suppose there are arcs $b$ and $b'$ of $\beta \cap Y$ such that the endpoints of $b$ separate the endpoints of $b'$ in $\alpha$ (see Figure 4.4a). This can happen only when $Y$ has positive genus. Let $\gamma$ be the boundary component in $Y$ of a regular neighbourhood of $\alpha \cup b \cup b'$. By the assumptions on $Y$, the curve $\gamma$ is essential and non-peripheral. Moreover, $\gamma$ is in the same class as $\alpha$ in $H_1(S, \partial S; \mathbb{Z})$ (with appropriate orientation), so is separating.

**Case 2.** Suppose there are no arcs of $\beta \cap Y$ arranged as in Case 1. Choose an arc $b$ of $\beta \cap Y$. Let $\gamma_1$ and $\gamma_2$ be the two components of a regular neighbourhood of $\alpha \cup b$ in $Y$ (Figure 4.4b). With appropriate orientations, $0 = [\alpha] = [\gamma_1] + [\gamma_2]$ in $H_1(S, \partial S; \mathbb{Z})$, so either both $\gamma_1$ and $\gamma_2$ are separating or neither is.

**2a.** Suppose that both curves are separating. It is possible that one of $\gamma_1$ or $\gamma_2$ could be peripheral (neither can be inessential by the assumption that $\alpha$ and $\beta$ are in minimal position). However, they cannot both be peripheral as otherwise $Y$ would be a planar subsurface containing only two components of $\partial S$, which contradicts the assumptions. Choose one of the two curves which is non-peripheral to be $\gamma$.

**2b.** Suppose that $\gamma_1$ and $\gamma_2$ are non-separating. Then there exists an essential arc $c$ in $Y$ with endpoints in $\alpha$ such that $c$ is disjoint from $b$ and the endpoints of $c$ separate the endpoints of $b$ in $\alpha$ (Figure 4.4c). Moreover, if $c$ intersects any other arc of $\beta$ then we can perform a surgery along the arc of $\beta$ to remove the intersection. Let $\gamma$ be the boundary component in $Y$ of a regular neighbourhood of $\alpha \cup b \cup c$. This is separating as in Case 1.

In each case, $\gamma$ satisfies the required conditions so we are done.




(a) Case 1. (b) Case 2a. (c) Case 2b.

Figure 4.4: The surgeries to produce the curve $\gamma$ in the different cases.

Now suppose that neither component of $S \setminus \alpha$ satisfies the conditions given for $Y$. That is, each component either has genus 1 and at most one component of $\partial S$ or genus 0 and at most two components of $\partial S$. By the assumptions on $S$, the only two possibilities are: $S$ is $S_{2,2}$ and both components are copies of $S_{1,2}$, or $S$ is $S_{1,3}$ with one component a copy of $S_{1,2}$ and the other a copy of $S_{0,3}$. Let $T$ be a component of $S \setminus \alpha$ which is homeomorphic to $S_{1,2}$. Every component of $T \setminus (\beta \cap T)$ contains some arc of $\alpha$, and one of the components contains the component of $\partial S$. Hence we can find an arc $c$ in $T$ joining the two boundary components ($\alpha$ and the component of $\partial S$) such that $c$ does not intersect $\beta$. Let $\alpha'$ be the boundary component of a regular neighbourhood of $c \cup \partial T$ which is essential and non-peripheral in $T$ (see Figure 4.5). The curve $\alpha'$ satisfies $i(\alpha', \alpha) = 0$ and $i(\alpha', \beta) \leqslant i(\alpha, \beta)$. Moreover, $S \setminus \alpha'$ has a component which satisfies the conditions above for $Y$, so we can construct $\gamma$ such that $i(\gamma, \alpha') = 0$ and $i(\gamma, \beta) < i(\alpha', \beta) \leqslant i(\alpha, \beta)$. Hence $\gamma$ is connected to $\alpha$ by construction and $\beta$ by the induction hypothesis, completing the proof.

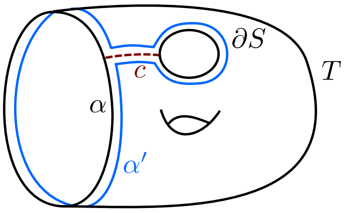



Figure 4.5: Finding a new curve $\alpha'$ when $\alpha$ does not have a complementary component satisfying the conditions for $Y$.

## 4.2 A graph of multicurves

In this section, we introduce a graph associated to a surface $S$ whose vertices are certain multicurves, and prove that it is hierarchically hyperbolic. We shall show in Section 4.3 that this graph is quasi-isometric to $\mathrm{Sep}(S)$.

### 4.2.1 Definition of $\mathcal{K}(S)$

Let $S$ be a surface as in Theorem 4.1.1. Below, we will define a graph $\mathcal{K}(S)$ whose vertices are multicurves which cut $S$ into subsurfaces which are not in the set $\mathfrak{X}$. In particular, every separating curve is a vertex of $\mathcal{K}(S)$. Also note that, since for any $X \in \mathfrak{X}$, any subsurface containing $X$ is also in $\mathfrak{X}$, the addition of a disjoint curve to any vertex of $\mathcal{K}(S)$ gives another vertex of $\mathcal{K}(S)$.

**Definition 4.2.1.** The graph $\mathcal{K}(S)$ has:

- a vertex for each multicurve $a$ in $S$ such that for every component of $S \setminus a$, there is a separating curve of $S$ disjoint from this component,
- an edge between vertices $a$ and $b$ if one of the following holds:
  1. $b$ is obtained either by adding a single curve to $a$ or by removing a single curve from $a$,
  2. $b$ is obtained by replacing a curve $\alpha$ in $a$ with a curve $\beta$, where the component of $S \setminus (a \setminus \alpha)$ containing $\alpha$ is in $\mathfrak{X}$ and is a copy of $S_{0,4}$, and $\alpha$ and $\beta$ intersect exactly twice.

The second type of edge can arise only when $S$ is $S_3$, $S_{3,1}$, $S_{2,2}$ or $S_{1,3}$, since these are the only cases where there are subsurfaces in $\mathfrak{X}$ which are copies of $S_{0,4}$. In principle, we could define a similar move in $S_{1,1}$ subsurfaces, but, since we assume that $S$ satisfies the hypotheses of Theorem 4.1.1, there is no subsurface in $\mathfrak{X}$ which is a copy of $S_{1,1}$. Note that we could more generally allow replacing a curve $\alpha$ in $a$ with a curve $\beta$, where the component $X$ of $S \setminus (a \setminus \alpha)$ containing $\alpha$ is in $\mathfrak{X}$, and $\alpha$ and $\beta$ are adjacent in the curve graph of $X$. When $X$ is a copy of $S_{0,4}$, then this gives the second type of edge. When $\xi(X) \geqslant 2$, this corresponds to two moves of the first type: adding a curve $\beta$ disjoint to all curves in $a$, then removing a curve $\alpha$. Hence including this move does not change the large scale geometry of the graph.

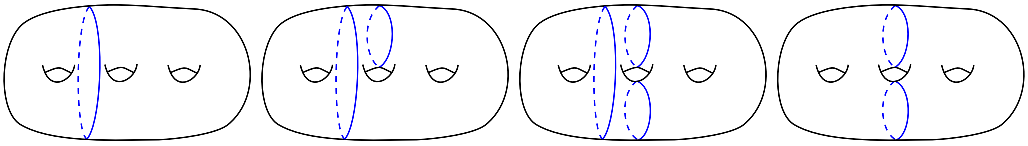


Figure 4.6: An example of a path in $\mathcal{K}(S_3)$.

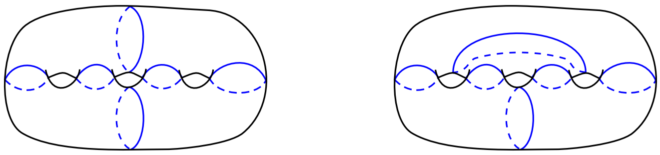


Figure 4.7: Another example of a path in $\mathcal{K}(S_3)$.

Note that connectedness of $\mathcal{K}(S)$ is implied by connectedness of the pants graph as follows. Every pants decomposition of $S$ is a vertex of $\mathcal{K}(S)$ and a pants move corresponds to either one or two moves in $\mathcal{K}(S)$. Moreover, each vertex of $\mathcal{K}(S)$ is connected to a pants decomposition by adding curves one by one. For a proof of connectedness of the pants graph, see [33]. From now on, for notational convenience, we shall treat $\mathcal{K}(S)$ as a discrete set of vertices equipped with the combinatorial metric induced from the graph.

**Proposition 4.2.2.** *Let $\mathfrak{Z}$ be the set of subsurfaces which every vertex of $\mathcal{K}(S)$ must intersect. Then $\mathfrak{Z} = \mathfrak{X}$.*

*Proof.* Firstly $\mathfrak{Z}$ is contained in $\mathfrak{X}$ since each separating curve is a vertex of $\mathcal{K}(S)$. Suppose $X$ is in $\mathfrak{X}$ and $a$ is a vertex of $\mathcal{K}(S)$. If $a$ does not cut $X$ then $X$ is contained in a single component of $S \setminus a$. But then $X$ has a separating curve in its complement, which contradicts that it is in $\mathfrak{X}$. $\square$

In Sections 4.2.2 and 4.2.3, we shall prove the following theorem.

**Theorem 4.2.3.** *Let $S$ be as in Theorem 4.1.1. The graph $\mathcal{K}(S)$ is a hierarchically hyperbolic space with respect to the set $\mathfrak{X}$.*

### 4.2.2 Verification of Axioms 1–8

As above, let $\mathfrak{X}$ be the set of subsurfaces which every vertex of $\mathcal{K}(S)$ (or equivalently of $\mathrm{Sep}(S)$) must intersect. We will verify that $\mathcal{K}(S)$ satisfies the axioms for hierarchical hyperbolicity (see Section 3.7.1) for $\mathfrak{S} = \mathfrak{X}$. For each $X \in \mathfrak{X}$, the $\delta$-hyperbolic space $\mathcal{C}(X)$ is the curve graph of $X$. The constant $\delta$ need not depend on the surface $S$, since curve graphs are uniformly hyperbolic [1, 12, 20, 35]. Most of the axioms follow easily from known results on subsurface projections. The only significant new work needed is the verification of Axiom 9. We reserve this for a separate section, and verify Axioms 1 to 8 below.

**1. Projections** Let $\pi_X : \mathcal{K}(S) \to 2^{\mathcal{C}(X)}$ be the usual subsurface projection (see Section 3.4). The image of a vertex is never empty since every vertex of $\mathcal{K}(S)$ intersects each $X$ in $\mathfrak{X}$. Let $a$ and $b$ be at distance 1 in $\mathcal{K}(S)$. Unless they are connected by a move in an $S_{0,4}$ subsurface, $a \cup b$ is a multicurve so its projection to any $\mathcal{C}(X)$ for $X \in \mathfrak{X}$ has diameter at most 2 by Proposition 3.4.1. Suppose $a$ and $b$ are connected by a move in a subsurface $X_\alpha \cong S_{0,4}$. If $X = X_\alpha$, then the projection of $a \cup b$ to $\mathcal{C}(X)$ is two adjacent curves and has diameter 1. Suppose $X \neq X_\alpha$. Since no subsurface of $X_\alpha$ can be in $X$, some curve of $\partial_S X_\alpha$ intersects $X$. This curve is disjoint from every curve of $a \cup b$ so the diameter of the projection is at most 4. Hence, the projection $\pi_X$ is 4-Lipschitz.

**2. Nesting.** The partial order on $\mathfrak{X}$ is inclusion of subsurfaces, with $X \sqsubseteq Y$ if $X$ is contained in $Y$. The unique $\sqsubseteq$-maximal element is $S$. If $X \sqsubsetneq Y$, then we can take $\pi_Y(X) = \partial_Y X \subset \mathcal{C}(Y)$, that is, all boundary curves of $X$ which are non-peripheral in $Y$. This has diameter at most 1 in $\mathcal{C}(Y)$ as the curves are pairwise disjoint. The projection $\pi^Y_X : \mathcal{C}(Y) \to 2^{\mathcal{C}(X)}$ is the subsurface projection from $\mathcal{C}(Y)$ to $2^{\mathcal{C}(X)}$.

**3. Orthogonality.** The orthogonality relation $\perp$ on $\mathfrak{X}$ is disjointness of subsurfaces. If $Z$ is disjoint from $Y$ then it is disjoint from any subsurface of $Y$. Suppose $X \in \mathfrak{X}$ and $Y \sqsubseteq X$. Then either no other subsurface of $X$ disjoint from $Y$ is in $\mathfrak{X}$, or the complement $Z = X \setminus Y$ is in $\mathfrak{X}$ and any $U \sqsubseteq X$ which is disjoint from $Y$ is nested in $Z$. Finally, if $X$ and $Y$ are disjoint then neither is nested in the other.

**4. Transversality and consistency.** Two subsurfaces $X$ and $Y$ in $\mathfrak{X}$ are transverse, $X \pitchfork Y$, if they are neither disjoint nor nested. If $X \pitchfork Y$, let $\pi_X(Y)$ be the subsurface projection of $\partial_S Y \subset \mathcal{C}(S)$ to $\mathcal{C}(X)$, and similarly for $\pi_Y(X)$. These each have diameter at most 2 by Proposition 3.4.1. By Behrstock's lemma (Theorem 4.3

of [5]), for each $S$ there exists $\kappa$ such that for any $X \pitchfork Y$ and any multicurve $a$ projecting to both (and hence any vertex $a$ of $\mathcal{K}(S)$),

$$\min\{d_X(\pi_X(a), \pi_X(Y)), d_Y(\pi_Y(a), \pi_Y(X))\} \leqslant \kappa.$$

For a more elementary proof due to Leininger, with a uniform value of $\kappa$, see Lemma 2.13 of [40]. Given $X \sqsubseteq Y$, and $a$ in $\mathcal{K}(S)$ consider

$$\min\{d_Y(\pi_Y(a), \pi_Y(X)), \operatorname{diam}_{\mathcal{C}(X)}(\pi_X(a) \cup \pi_X^Y(\pi_Y(a)))\}.$$

The second term compares projecting $a$ directly to $\mathcal{C}(X)$ from $\mathcal{K}(S)$ and projecting $a$ first to $\mathcal{C}(Y)$ and then to $\mathcal{C}(X)$. This gives the same result, so this quantity is $\operatorname{diam}_{\mathcal{C}(X)}(\pi_X(a)) \leqslant 2$. Also, if $X \sqsubseteq Y$, then the union of their boundary components is a multicurve in $\mathcal{C}(S)$, so for any $Z \in \mathfrak{X}$ such that $Y \sqsubsetneq Z$ or $Y \pitchfork Z$ and $X \not\perp Z$, $d_Z(\pi_Z(X), \pi_Z(Y)) \leqslant 2$.

**5. Finite complexity.** The length of a chain of nested subsurfaces in $\mathfrak{X}$ is bounded above by $\xi(S)$.

**6. Large links.** Let $X \in \mathfrak{X}$ and $a, b \in \mathcal{K}(S)$, with $R = d_X(a, b) + 1$. Assume for now that $\xi(X) \geqslant 2$. Let $\gamma_1, \gamma_2, \ldots, \gamma_{R-1}, \gamma_R$ be a geodesic in $\mathcal{C}(X)$, where $\gamma_1 \in \pi_X(a)$ and $\gamma_R \in \pi_X(b)$. For each $1 \leqslant i \leqslant R$, let $Y_i$ be the component of $X \setminus \gamma_i$ containing the adjacent curves of the geodesic. Note that $Y_i$ is not necessarily in $\mathfrak{X}$. Suppose $Y \in \mathfrak{X}$ satisfies $Y \sqsubsetneq X$ and $d_Y(a, b) > M$, where $M$ is the constant of Theorem 3.1 of [42] (Bounded Geodesic Image; see also Axiom 7 below for more detail). The Bounded Geodesic Image Theorem implies that, in this case, some $\gamma_i$ does not intersect $Y$. Hence $Y$ is contained in a single component of $S \setminus \gamma_i$. Suppose that this component is not $Y_i$. Then the adjacent curves to $\gamma_i$ in the geodesic also do not cut $Y$. Since $S \setminus Y_i$ is contained in $Y_{i-1}$ or $Y_{i+1}$, so too is $Y$. Hence, $Y$ is contained in some $Y_i$. We also need to check that this $Y_i$ is in $\mathfrak{X}$. This follows from the fact that $Y$ is in $\mathfrak{X}$, and hence so is any subsurface containing $Y$. We include only those $Y_i$ which are in $\mathfrak{X}$ in the list. If there are no subsurfaces of $\mathfrak{X}$ properly nested in $X$, and, in particular, if $X \cong S_{0,4}$, then trivially $d_Y(a, b) \leqslant M$ for every $Y \in \mathfrak{X}$ with $Y \sqsubsetneq X$. Finally, for each $i$, we have $d_X(\pi_X(a), \pi_X(Y_i)) = d_X(\pi_X(a), \pi_X(\gamma_i)) \leqslant R$.

**7. Bounded geodesic image.** By Theorem 3.1 of [42], there exists $M$ so that for all $Y \sqsubsetneq X$, and any geodesic $g$ in $\mathcal{C}(X)$, either $\operatorname{diam}_{\mathcal{C}(Y)}(g) \leqslant M$ or some vertex $\gamma$ of $g$ does not intersect $Y$. If $\gamma$ is disjoint from $Y$, then it is adjacent in $\mathcal{C}(X)$ to $\pi_X(Y) = \partial_X Y$. Hence, if $\operatorname{diam}_{\mathcal{C}(Y)}(g) > M$, then $g \cap N_{\mathcal{C}(X)}(\pi_X(Y), 1) \neq \varnothing$, and so the conditions of this axiom are satisfied for $E = M$. For a proof that the constant $M$ does not depend on the surface $S$, see [59].

**8. Partial realisation.** Any set of pairwise disjoint subsurfaces in $\mathfrak{X}$ contains at most two elements (at most one if $S$ has at least three boundary components). First suppose the set contains only one element $X_1$. Let $\gamma_1$ be a curve in $X_1$. Consider the multicurve $\partial_S X_1 \cup \gamma_1$. We may complete this to a vertex of $\mathcal{K}(S)$ by, for example, adding curves to obtain a pants decomposition of $S$. Firstly, the projection of $a$ to $X_1$ is a multicurve containing $\gamma_1$, so $d_{X_1}(\pi_{X_1}(a), \gamma_1) \leqslant 1$. Let $X$ be a subsurface of $S$ containing $X_1$. Then $d_X(\pi_X(a), \pi_X(X_1)) \leqslant 2$, by Proposition 3.4.1, since $a$ contains $\partial_S X_1$. Let $Y \in \mathfrak{X}$ be transverse to $X_1$. Then similarly $d_Y(\pi_Y(a), \pi_Y(X_1)) \leqslant 2$. Now suppose $X_1$ and $X_2$ are distinct and disjoint subsurfaces in $\mathfrak{X}$. Let $\gamma_j$ be a curve in $X_j$ for each $j$. Again, there exists $a$ in $\mathcal{K}(S)$ containing $\gamma_1$, $\gamma_2$, $\partial_S X_1$ and $\partial_S X_2$. Moreover, as before, for each $j$, $d_{X_j}(\pi_{X_j}(a), \gamma_j) \leqslant 1$, $d_X(\pi_X(a), \pi_X(X_j)) \leqslant 2$ for every $X$ containing $X_j$, and $d_Y(\pi_Y(a), \pi_Y(X_j)) \leqslant 2$ for every $Y$ transverse to $X_j$.

We remark that all of the above constants, apart from the complexity, may be taken to be independent of the surface $S$. Our proof below that Axiom 9 holds gives constants which do depend on the surface $S$ and are probably far from optimal. It would be interesting to consider how far they can be improved. The quasi-isometry constants in Section 4.3 also *a priori* depend on the surface.

### 4.2.3 Verification of Axiom 9

The most significant part of the proof of Theorem 4.2.3 is the verification of the final axiom. For brevity of notation, we will now suppress the projection maps when considering distances and diameters for subsurface projections.

**Proposition 4.2.4.** *Let $S$ satisfy the hypotheses of Theorem 4.1.1. For every $K$, there exists $K'$, depending only on $K$ and $\xi(S)$, such that if $a$ and $b$ are two vertices of $\mathcal{K}(S)$, and if $d_X(a,b) \leqslant K$ for every subsurface $X$ in $\mathfrak{X}$, then $d_{\mathcal{K}(S)}(a,b) \leqslant K'$.*

In order to prove this, we make use of a combinatorial construction based on that described in Section 10 of [14]. This will give us a way of representing a sequence of multicurves in $S$. We shall construct this sequence inductively so that eventually it will be a path in $\mathcal{K}(S)$. We remark that this method is also related to the hierarchy machinery of [42].

We shall consider the product $S \times I$, for a non-trivial closed interval $I$. We consider $S$ to be the *horizontal* direction and $I$ to be the *vertical* direction. We have a vertical projection $S \times I \to S$ and a horizontal projection $S \times I \to I$. When we denote a subset of $S \times I$ by $A_1 \times A_2$, $A_1$ will be a subset of the horizontal factor, $S$, and $A_2$ of the vertical factor, $I$. To ensure that curves in $S$ are pairwise in minimal position, we will fix a hyperbolic structure on $S$ with totally geodesic boundary and take the geodesic representative of each isotopy class of curves.

**Definition 4.2.5.** A *vertical annulus* in $S \times I$ is a product $\gamma \times I_\gamma$, where $\gamma$ is a curve in $S$ and $I_\gamma$ is a non-trivial closed subinterval of $I$. The curve $\gamma$ is the *base curve* of the annulus.

**Definition 4.2.6.** An *annulus system* $W$ in $S \times I$ is a finite collection of disjoint vertical annuli. An annulus system $W$ is *generic* if whenever $\gamma_1 \times I_1$ and $\gamma_2 \times I_2$ are two distinct annuli in $W$, we have $\partial I_1 \cap \partial I_2 \subseteq \partial I$.

We denote $S \times \{t\}$ by $S_t$ and $W \cap S_t$ by $W_t$. Each $W_t$ is a (possibly empty) multicurve, and there is a discrete set of points in $I$ where the multicurve $W_t$ changes. Hence the annulus system is a way of recording a sequence of multicurves in $S$.

**Definition 4.2.7.** Let $\xi(S) \geqslant 2$. A *tight geodesic* in $\mathcal{C}(S)$ between curves $\gamma$ and $\gamma'$ is a sequence $\gamma = v_0, v_1, \ldots, v_{n-1}, v_n = \gamma'$, where:

- each $v_i$ is a multicurve in $S$,
- for any $i \neq j$ and any curves $\gamma_i \in v_i$, $\gamma_j \in v_j$, $d_S(\gamma_i, \gamma_j) = |i - j|$,
- for each $1 \leqslant i \leqslant n-1$, $v_i$ is the boundary multicurve of the subsurface spanned by $v_{i-1}$ and $v_{i+1}$ (excluding any components of $\partial S$).

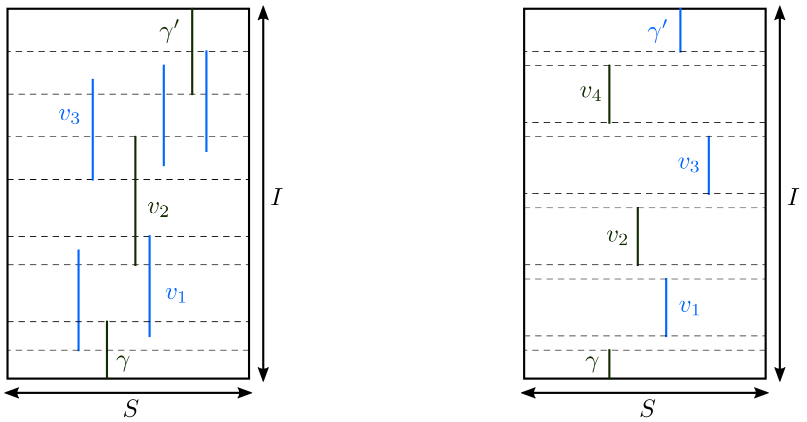
If $\xi(S) = 1$, then a tight geodesic is an ordinary geodesic in $\mathcal{C}(S)$.

This definition comes from [42], although the tight geodesics of [42] are equipped with some additional data which will not be relevant here. A tight geodesic can be realised as an annulus system as follows.

**Definition 4.2.8.** A *tight ladder* in $S \times I$ is a generic annulus system $W$ so that:

- there exists a tight geodesic $v_0, v_1, \ldots, v_{n-1}, v_n$ in $\mathcal{C}(S)$ so that the curves appearing in the tight geodesic correspond exactly to the base curves of the annuli in $W$,
- for two annuli $\gamma \times I_\gamma$ and $\delta \times I_\delta$ in $W$, the intervals $I_\gamma$ and $I_\delta$ intersect if and only if $\gamma$ and $\delta$ are disjoint,
- there exist $t_0 < t_1 < \cdots < t_{n-1} < t_n$ in $I$ such that for each $i$ the multicurve $W_{t_i} = v_i$.

In the case where $\xi(S) \geqslant 2$, this corresponds to moving from $v_i$ to $v_{i+1}$ by adding in the curves of $v_{i+1}$ one at a time then removing the curves of $v_i$ one at a time (Figure 4.8a). In the case where $\xi(S) = 1$, this corresponds to moving from $v_i$ to $v_{i+1}$ by removing the curve $v_i$ then adding in the curve $v_{i+1}$ after a vertical interval with no annuli (Figure 4.8b).




(a) Complexity $\xi(S) \geqslant 2$. (b) Complexity $\xi(S) = 1$.

Figure 4.8: Illustrations of tight ladders in $S \times I$.

From now on, we will assume that $S$ satisfies the hypotheses of Theorem 4.1.1.

**Definition 4.2.9.** Let $t \in I$, and let $X$ be a component of $S_t \setminus W_t$. Let $J \subseteq I$ be the maximal interval containing $t$ such that $X$ is a component of $S_s \setminus W_s$ for every $s \in J$. The product $X \times \bar{J}$ is a *brick* of $W$. The surface $X$ is the *base surface* of the brick.

We remark that this differs slightly from the definition of "brick" in [14]. Note that the interiors of any two distinct bricks are disjoint, and that we may decompose $S \times I$ as a union of regular neighbourhoods of all bricks of $W$ (recall that when we remove a multicurve $a$ from $S$, we also remove a regular open neighbourhood of $a$). In order to obtain a path in $\mathcal{K}(S)$, we want to decompose $S \times I$ into bricks whose base surfaces are not in $\mathfrak{X}$.

**Definition 4.2.10.** A brick $X \times [s, t]$ is *small* if one of the following holds.

(Type 1) The base surface $X$ is not in $\mathfrak{X}$.

(Type 2) The base surface $X$ is a copy of $S_{0,4}$ and is in $\mathfrak{X}$. Moreover, $W_s$ and $W_t$ each intersect $X$ in an essential non-peripheral curve, and the two curves are adjacent in $\mathcal{C}(X)$.

Notice that a generic annulus system $W$ where every brick is small realises a path in $\mathcal{K}(S)$, as follows. First assume there are no copies of $S_{0,4}$ in $\mathfrak{X}$. Consider the multicurves $W_t$ for $t \in I$. These change precisely at the points in the interior of $I$ which are the endpoints of horizontal projections of annuli in $W$. Let $P$ denote this set of points. Let $I_0, \ldots, I_n$ be the components of $I \setminus P$ in the order in which they appear in $I$, and for each $0 \leqslant j \leqslant n$ pick any $t_j$ from $I_j$. Let $a_j$ be the multicurve $W_{t_j}$. The sequence $a_0, \ldots, a_n$ is a path in $\mathcal{K}(S)$.

In the case where there are copies of $S_{0,4}$ in $\mathfrak{X}$, we place an additional restriction on a generic annulus system, requiring that whenever we have a Type 2 small brick, the endpoints of its horizontal projection to $I$ are consecutive points of $P$. This can be achieved by appropriate isotopies. Again, let $W$ be a generic annulus system where every brick is small. Construct the sequence of curves $a_j$ as above and suppose that, for some $j$, $S \setminus a_j$ has a component $X$ which is an $S_{0,4}$ subsurface in $\mathfrak{X}$ (and hence $a_j$ is not a vertex of $\mathcal{K}(S)$). Then by the restriction on the endpoints of the horizontal projection of a Type 2 small brick, $X$ is not a component of $S \setminus a_{j-1}$ or $S \setminus a_{j+1}$, nor is any other $S_{0,4}$ subsurface in $\mathfrak{X}$. Then $a_{j-1}$ and $a_{j+1}$ are adjacent vertices of $\mathcal{K}(S)$. Hence we obtain a path in $\mathcal{K}(S)$ as for the previous case except that we remove any multicurves in the sequence $a_0, \ldots, a_n$ which are not vertices of $\mathcal{K}(S)$.

**Definition 4.2.11.** The *$\mathcal{K}$-complexity* of an annulus system $W$ is $(N_{\xi(S)}, N_{\xi(S)-1}, \ldots, N_1)$, where, for each $i$, $N_i$ is the total number of non-small bricks of $W$ whose base surface is a subsurface in $\mathfrak{X}$ of complexity $i$. We give this the lexicographical ordering.

Since there are no subsurfaces in $\mathfrak{X}$ of complexity less than 1, the $\mathcal{K}$-complexity is $(0, 0, \ldots, 0)$ precisely when every brick is small.

We now begin the proof of Proposition 4.2.4. Let $I = [0, 1]$. We shall construct a generic annulus system in $S \times I$, with $\mathcal{K}$-complexity $(0, 0, \ldots, 0)$, which realises a path in $\mathcal{K}(S)$ from $a$ to $b$, and show that the length of this path is bounded in terms of $K$ and $\xi(S)$.

We construct the annulus system inductively. We start by choosing distinct points $t_\alpha \in (0, \frac{1}{2})$ for each curve $\alpha$ of $a$ and $t_\beta \in (\frac{1}{2}, 1)$ for each curve $\beta$ of $b$ and defining an annulus system $W^{(0)} = \bigcup_\alpha (\alpha \times [0, t_\alpha]) \cup \bigcup_\beta (\beta \times [t_\beta, 1])$.

We will describe below the procedure for constructing a new annulus system $W^{(k+1)}$ from $W^{(k)}$, where the first annulus system $W^{(0)}$ is as defined above. We shall do this in such a way that each annulus system interpolates between $a$ and $b$ (in fact, $W^{(k+1)}$ contains $W^{(k)}$), and such that the $\mathcal{K}$-complexity of $W^{(k+1)}$ is strictly less than that of $W^{(k)}$. This process will eventually terminate with an annulus system with $\mathcal{K}$-complexity $(0, 0, \ldots, 0)$.

Suppose we have constructed a generic annulus system $W^{(k)}$. We will describe how to construct the next stage $W^{(k+1)}$; see Figure 4.9 for an illustration. Consider the bricks of $W^{(k)}$. If every brick is small, then the $\mathcal{K}$-complexity of $W^{(k)}$ is $(0, \ldots, 0)$ and we are done. Suppose this is not the case, and choose a brick $Y \times [t_-, t_+]$, where $Y$ is in $\mathfrak{X}$ and has maximal complexity among such bricks. (Note that *a priori* the same subsurface $Y$ might appear as the base surface of more than one brick.) Decreasing past $t_-$ and increasing past $t_+$, the components of $S_t \setminus W_t^{(k)}$ change to not include $Y$. Since $Y$ has maximal complexity among base surfaces of $W^{(k)}$ in $\mathfrak{X}$, it is not a proper subsurface of any component of $S_t \setminus W_t^{(k)}$ for any $t \in I$. Hence, the intersection of $W_{t_-}^{(k)}$ and of $W_{t_+}^{(k)}$ with $Y$ must be non-empty, and, since $W^{(k)}$ is generic, it is in each case a single curve, which we call $\gamma_-$ and $\gamma_+$ respectively. Slightly extend $[t_-, t_+]$ on each side to $J = [t_- - \epsilon, t_+ + \epsilon]$ so that the subset $Y \times J$ now contains vertical annuli corresponding to each of these curves but still intersects no other annuli. We may consider annulus systems in $Y \times J$ as for $S \times I$. Add a tight ladder in $Y \times J$, corresponding to a tight geodesic in $\mathcal{C}(Y)$ from $\gamma_-$ to $\gamma_+$, arranging that the resulting annulus system in $S \times I$ is generic by slightly moving the endpoints of intervals if necessary. The annulus system $W^{(k+1)}$ is the union of $W^{(k)}$ and the tight ladder in $Y \times J$. Notice that the $\mathcal{K}$-complexity of $W^{(k+1)}$ is strictly less than that of $W^{(k)}$.

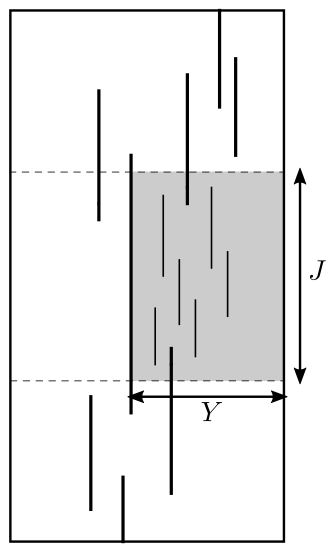



Figure 4.9: Constructing $W^{(k+1)}$ from $W^{(k)}$ by adding a tight ladder in a brick $Y \times J$.

At each stage, we add a tight ladder $v_0, v_1, \ldots, v_{n-1}, v_n$ in some brick, $Y \times J$, increasing the length of the sequence of multicurves determined by the annulus system, where these multicurves are not yet necessarily vertices of $\mathcal{K}(S)$. Let us consider the maximal increase in the length of this sequence. Let $Q$ be the set of points in the interior of $J$ corresponding to the endpoints of the horizontal projections of bricks to $J$. First suppose $\xi(Y) \geqslant 2$. The transition from $v_i$ to $v_{i+1}$ gives a point of $Q$ for every curve in $v_i$ and every curve in $v_{i+1}$, so $|Q| = (|v_0| + |v_1|) + (|v_1| + |v_2|) + \cdots + (|v_{n-1}| + |v_n|) \leqslant n\xi(Y)$. Now suppose $\xi(Y) = 1$. Then the number of points of $Q$ is $2n = 2n\xi(Y)$. Hence between $W^{(k)}$ and $W^{(k+1)}$, when we add a tight ladder of length $n$ in a brick $Y \times J$, we add at most $2n\xi(Y)$ to the length of the corresponding sequences of curves.

The length of the tight ladder we add between $W^{(k)}$ and $W^{(k+1)}$ is equal to $d_Y(\gamma_-, \gamma_+)$. We now show that this quantity is bounded above in terms of $k$ and $K$.

**Claim 4.2.12.** Let $\Gamma^{(k)}$ be the set of the base curves of all annuli in $W^{(k)}$ and $K$ as in the statement of Proposition 4.2.4. Then $\mathrm{diam}_{\mathcal{C}(X)}(\pi_X(\Gamma^{(k)})) \leqslant 3^k K$ for each $X \in \mathfrak{X}$.

We prove this by an induction on $k$. The base case is when $k = 0$ and holds since, by hypothesis, $\mathrm{diam}_{\mathcal{C}(X)}(a \cup b) \leqslant K$ for every $X \in \mathfrak{X}$. Suppose at stage $k-1$ the projection has diameter at most $3^{k-1}K$. At stage $k$, we add a tight geodesic $v_0, v_1, \ldots, v_{n-1}, v_n$ in $\mathcal{C}(Y)$ for some $Y \in \mathfrak{X}$, where $v_0$ and $v_n$ are curves which already appear as base curves in $W^{(k-1)}$. By the induction hypothesis, $n = d_Y(v_0, v_n) \leqslant 3^{k-1}K$. There are several cases depending on how $X$ and $Y$ intersect.

*Case 1:* $X$ is disjoint from $Y$. Then none of the curves added in $Y$ contributes to the projection to $X$ so the diameter is unchanged.

*Case 2:* $X$ intersects $Y$ and is not nested in $Y$. Then there is a curve $\delta$ in $\partial_S Y$ which intersects $X$ non-trivially. Such a curve is also a base curve in $W^{(k-1)}$. Every curve added in $Y$ is disjoint from $\delta$. Hence every curve added either does not intersect $X$ so does not change the projection to $\mathcal{C}(X)$, or projects to a curve at distance at most 2 from $\pi_X(\delta)$. Hence, the diameter of the projection increases by at most 4.

*Case 3:* $X$ is nested in $Y$. Suppose that some multicurves $v_p$ and $v_q$ in the tight geodesic do not intersect $X$, for $p < q$. Then there is a curve in $X$ which intersects neither, so we have $d_Y(v_p, v_q) \leqslant 2$, and $q \leqslant p + 2$ by the definition of a tight geodesic. Moreover, if $q = p + 2$ then $v_{p+1}$ also does not intersect $X$ since it is the boundary of the subsurface spanned by $v_p$ and $v_q$. Hence, any multicurves in the geodesic which do not cut $X$ are consecutive terms. Let $v_p$ and $v_q$ be respectively the first and last terms which do not intersect $X$. Suppose $p > 0$ and $q < n$. Then the increase in diameter between $\pi_X(\Gamma^{(k-1)})$ and $\pi_X(\Gamma^{(k)})$ is at most the sum of the maximal possible distances from $\pi_X(v_0)$ to $\pi_X(v_{p-1})$ and from $\pi_X(v_{q+1})$ to $\pi_X(v_n)$. By Proposition 3.4.1, and the induction hypothesis,

$$\operatorname{diam}_{\mathcal{C}(X)}(\Gamma^{(k)}) \leqslant 3^{k-1}K + 2(p-1) + 2(n-(q+1)) \leqslant 3^{k-1}K + 2 \cdot 3^{k-1}K = 3^k K.$$

Similarly, if $p = 0$ or $q = n$ then we have only one of the terms $2(p-1)$ or $2(n-(q-1))$, and again $\operatorname{diam}_{\mathcal{C}(X)}(\Gamma^{(k)}) \leqslant 3^k K$. If every term in the tight geodesic cuts $X$ then the increase in diameter from $W^{(k-1)}$ is bounded above by the maximal distance from $v_0$ or $v_n$ to the middle term. In any case, $\operatorname{diam}_{\mathcal{C}(X)}(\Gamma^{(k)}) \leqslant 3^k K$. This completes the proof of Claim 4.2.12.

In order to find an upper bound on the length of the final path in $\mathcal{K}(S)$, we will find upper bounds on the length of the sequence of curves at certain stages of the induction. For each $1 \leqslant i \leqslant \xi(S)$, let $k_i$ be minimal such that $N_j^{(k_i)} = 0$ for all $i \leqslant j \leqslant \xi(S)$. In particular $k_{\xi(S)} \leqslant k_{\xi(S)-1} \leqslant \cdots \leqslant k_1$, and $k_1$ is the stage where the $\mathcal{K}$-complexity of the annulus system reaches $(0, 0, \ldots, 0)$. For $1 \leqslant i \leqslant \xi(S)$, define $T_i$ by $T_{\xi(S)} = (2K+2)\xi(S)$ and $T_i = T_{i+1} + Ki3^{1+2T_{\xi(S)}+\cdots+2T_{i+1}}$ for $1 \leqslant i \leqslant \xi(S) - 1$, and define $L_i$ by $L_{\xi(S)} = 1$ and $L_i = 1 + 2T_{\xi(S)} + \cdots + 2T_{i+1}$ for $1 \leqslant i \leqslant \xi(S) - 1$.

**Claim 4.2.13.** For each $1 \leqslant i \leqslant \xi(S)$, $k_i \leqslant L_i$ and the length of the sequence of curves corresponding to $W^{(k_i)}$ is at most $T_i$.

We shall prove this by a reverse induction on $i$. We start with the annulus system $W^{(0)}$ defined above. Between $W^{(0)}$ and $W^{(1)}$, we add a tight ladder in the maximal complexity brick, the length of which is at most $K$. There is now no brick of complexity $\xi(S)$, so $k_{\xi(S)} = 1 = L_{\xi(S)}$. The length of the sequence of multicurves given by $W_t^{(1)}$ is at most $|a| + |b| + 2K\xi(S) \leqslant (2K+2)\xi(S) = T_{\xi(S)}$.

Now assume for induction that $k_{i+1} \leqslant L_{i+1}$ and that the length of the sequence of multicurves given by $W_t^{(k_{i+1})}$ is at most $T_{i+1}$. If there are no bricks of complexity $i$, then $k_i = k_{i+1}$ and we are done, so suppose there is at least one. For each multicurve, there are at most two complementary components which are in $\mathfrak{X}$, since a set of pairwise disjoint subsurfaces in $\mathfrak{X}$ has cardinality at most 2 (see Section 4.1.2). Hence, $N_i^{(k_{i+1})} \leqslant 2T_{i+1}$. The maximal complexity is now $i$ so we add tight ladders in bricks of complexity $i$ until there are no more. We will need to do this at most $2T_{i+1}$ times, so $k_i \leqslant k_{i+1} + 2T_{i+1} \leqslant L_{i+1} + 2T_{i+1} = L_i$. The length of the tight ladder we add between $W^{(k)}$ and $W^{(k+1)}$ is at most $3^k K$, by Claim 4.2.12, and so adds at most $2Ki3^k$ to the length of the sequence of multicurves. Hence, in total, between $W^{(k_{i+1})}$ and $W^{(k_i)}$, we add at most the following to the length of the sequence of multicurves:

$$
\begin{aligned}
&2Ki(3^{k_{i+1}} + 3^{k_{i+1}+1} + \cdots + 3^{k_{i+1}+2T_{i+1}-1}) = 2Ki3^{k_{i+1}}(1 + 3 + \cdots + 3^{2T_{i+1}-1}) \\
&\quad = 2Ki3^{k_{i+1}} \frac{3^{2T_{i+1}} - 1}{3 - 1} \leqslant Ki3^{k_{i+1}+2T_{i+1}} \leqslant Ki3^{L_{i+1}+2T_{i+1}} = Ki3^{1+2T_{\xi(S)}+\cdots+2T_{i+1}}.
\end{aligned}
$$

Therefore, the length of the sequence of multicurves given by $W_t^{(k_i)}$ is at most $T_{i+1} + Ki3^{1+2T_{\xi(S)}+\cdots+2T_{i+1}} = T_i$, proving Claim 4.2.13.

In particular, the length of the sequence of multicurves corresponding to $W_t^{(k_1)}$ is at most $T_1$, which is a function of $K$ and $\xi(S)$. At this stage, the $\mathcal{K}$-complexity is $(0, 0, \ldots, 0)$, so this sequence of multicurves in fact gives a path in $\mathcal{K}(S)$ joining $a$ and $b$. Taking $K' = T_1$, this completes the proof of Proposition 4.2.4, and hence also of Theorem 4.2.3.

## 4.3 The separating curve graph

We now relate $\mathcal{K}(S)$ to $\mathrm{Sep}(S)$ to prove Theorem 4.1.1. Since every separating curve is a vertex of $\mathcal{K}(S)$, there is a natural inclusion $\phi\colon \mathrm{Sep}(S) \to \mathcal{K}(S)$ defined by $\phi(\alpha) = \{\alpha\}$ for every separating curve $\alpha$. Again, we are considering $\mathrm{Sep}(S)$ and $\mathcal{K}(S)$ as discrete sets of vertices with the induced combinatorial metric.

**Proposition 4.3.1.** *Let $S$ be as in Theorem 4.1.1. Then the inclusion $\phi\colon$ Sep$(S) \to \mathcal{K}(S)$ is a quasi-isometry.*

We first make the observation that in order to find an upper bound on the distance between two vertices in $\mathrm{Sep}(S)$ or in $\mathcal{K}(S)$ it is sufficient to bound their intersection number. To see this, fix some $n$. For each of the two graphs, up to the action of the mapping class group, there are only finitely many pairs of vertices intersecting at most $n$ times. Each graph has an isometric action of $\mathrm{MCG}(S)$, so we can take any pair of vertices intersecting at most $n$ times to one of these finitely many pairs without changing the distance between the vertices. Moreover, each of these graphs is connected so there is a maximal distance between the vertices in any such pair, which depends only on $n$ and the surface $S$.

The most substantial part of the proof of Proposition 4.3.1 is to show that the distance between two separating curves in $\mathcal{K}(S)$ is bounded below by a linear function of their distance in $\mathrm{Sep}(S)$. To prove this, we associate a bounded diameter subset of $\mathrm{Sep}(S)$ to each vertex of $\mathcal{K}(S)$. Let $N$ be a constant such that for each vertex $a$ of $\mathcal{K}(S)$ there is some separating curve which intersects $a$ at most $N$ times. Such an $N$ exists since, up to the action of the mapping class group, there are only finitely many vertices of $\mathcal{K}(S)$. Fixing some separating curve $\gamma$, we can take $N$ to be the maximal number of times $\gamma$ intersects any of this finite list of vertices. Given a vertex $a$ of $\mathcal{K}(S)$, define $C_a = \{\gamma \in \mathrm{Sep}(S) \mid i(\gamma, a) \leqslant N\}$. This is non-empty by construction.

**Lemma 4.3.2.** *There exists $N'$, depending only on $N$ and $\xi(S)$, such that the diameter of $C_a$ in Sep$(S)$ is at most $N'$.*

*Proof.* Note that throughout "separating curve" will refer to a curve that is separating in $S$ even when the curve is being chosen to be contained in a particular subsurface. Let $a$ be a vertex of $\mathcal{K}(S)$ and $\beta$, $\beta'$ two separating curves each intersecting $a$ at most $N$ times. If some curve of $a$ is separating, then we have a separating curve which intersects both $\beta$ and $\beta'$ at most $N$ times and this gives a bound on the distance between $\beta$ and $\beta'$ depending only on $N$ and $S$. Assume, therefore, that no curve of $a$ is separating. We shall use the fact that (by definition of $\mathcal{K}(S)$) for each component $Y$ of $S \setminus a$, there is a separating curve $\gamma$ disjoint from $Y$. Furthermore, since $\beta$ intersects $a$ at most $N$ times, up to the action of the mapping class group there are only finitely many possibilities for $\beta \cap (S \setminus Y)$, which is a collection of at most $N$ arcs (or a single curve) in $S \setminus Y$. Hence, we can choose $\gamma$ to have bounded intersection with $\beta$, where the bound depends only on $N$ and $S$. The same argument applies for $\beta'$.

We shall split the proof into several cases, observing that if a vertex $a'$ of $\mathcal{K}(S)$ is obtained by removing curves from another vertex $a$, then $C_a$ is a subset of $C_{a'}$ and hence the diameter of $C_a$ in $\mathrm{Sep}(S)$ is bounded above by the diameter of $C_{a'}$. For every vertex $a$ of $\mathcal{K}(S)$, either $a$ will fit into one of the first four cases below, or there will exist another vertex $a'$ of $\mathcal{K}(S)$ which is obtained from $a$ by removing curves and which fits into one of the cases.

**Case 1.** First suppose that $S \setminus a$ has only two components $Y_1$ and $Y_2$ (by definition of $\mathcal{K}(S)$, there cannot be only one component). Choose a separating curve $\gamma_1$ in $Y_2 = S \setminus Y_1$ such that $\gamma_1$ has bounded intersection with $\beta$, and choose $\gamma_2$ in $Y_1$ with bounded intersection with $\beta'$. Since $\gamma_1$ and $\gamma_2$ are disjoint, this gives a bound on the distance in $\mathrm{Sep}(S)$ between $\beta$ and $\beta'$ depending on $N$ and $\xi(S)$.

**Case 2.** Now suppose that $S \setminus a$ has more than two components, and that there is some component $Y_1$ such that $S \setminus Y_1$ is disconnected. Choose a separating curve $\gamma_1$ in one of the components $Z$ of $S \setminus Y_1$ such that $\gamma_1$ has bounded intersection with $\beta$. Note that $S \setminus Z$ is connected. Suppose that $Z$ is in $\mathfrak{X}$. Then $S \setminus Z$ is planar and attached to $Z$ by either all or all but one of its boundary components, otherwise $S \setminus Z$ would contain a separating curve. However, then no subsurface of $S \setminus Z$ could have a disconnected complement in $S$, contradicting that $Y_1$ has this property. Therefore, $Z$ is not in $\mathfrak{X}$. Then there is a separating curve $\gamma_2$ in $S \setminus Z$, and we can choose $\gamma_2$ to have bounded intersection with $\beta'$. As above, this gives a bound on $d_{\mathrm{Sep}(S)}(\beta, \beta')$ depending on $N$ and $\xi(S)$.

**Case 3.** Suppose that $S \setminus a$ has three components, $Y_1$, $Y_2$, $Y_3$, and that the complement of each component in $S$ is connected. We will construct a sequence $\beta, \gamma_1, \gamma_2, \gamma_3, \gamma_4, \beta'$ of separating curves of $S$, such that the intersection number between consecutive curves is bounded by a constant depending only on $N$ and $\xi(S)$. Since $\beta$ has bounded intersection with $a$, there are only finitely many possibilities for $(\beta \cup a) \cap (S \setminus Y_1)$ up to the action of the mapping class group. Hence we can choose a separating curve $\gamma_1$ in $S \setminus Y_1$ whose intersection with both $\beta$ and $a$ is bounded in terms of $N$ and $\xi(S)$. Since $\gamma_1$ has bounded intersection with $a$, up to the action of the mapping class group there are only finitely many possibilities for $(\gamma_1 \cup a) \cap (S \setminus Y_2)$. Hence, we can find a separating curve $\gamma_2$ in $S \setminus Y_2$ which has bounded intersection with $\gamma_1$ and with $a$. Now choose a separating curve $\gamma_4$ in $S \setminus Y_1$ such that $\gamma_4$ has bounded intersection with $\beta'$ and with $a$. The curve $\gamma_2$ is contained in $Y_3 \cup Y_1$ and $\gamma_4$ is contained in $Y_3 \cup Y_2$ (so they do not intersect in $S \setminus Y_3$) and both have bounded intersection with $a$. Hence, up to the action of the mapping class group, there are only finitely many possibilities for $(\gamma_2 \cup \gamma_4) \cap (S \setminus Y_3)$. We can therefore find a separating curve $\gamma_3$ in $S \setminus Y_3$ which intersects both $\gamma_2$ and $\gamma_4$ a bounded number of times. This once more gives a bound on $d_{\mathrm{Sep}(S)}(\beta, \beta')$ depending only on $N$ and $\xi(S)$.

**Case 4.** Suppose that $S \setminus a$ has four components, $Y_1$, $Y_2$, $Y_3$, $Y_4$, and that the complement of each of these components in $S$ is connected.

We can represent how the four components are connected by dual graphs. We avoid loops and multiple edges and instead put a single edge between distinct vertices if the components they represent meet along a multicurve. The possible configurations are precisely the 2-vertex-connected simple graphs on four vertices and are shown (up to symmetries) in Figure 4.10. The marked vertices will be explained shortly.

Note that if the union of two components is connected and is not in $\mathfrak{X}$ then we could reduce to the case of three components by removing a curve of $a$ which meets both components, while staying in the vertex set of $\mathcal{K}(S)$. We will hence suppose that for any pair of components whose union in $S$ is connected, that union is a subsurface

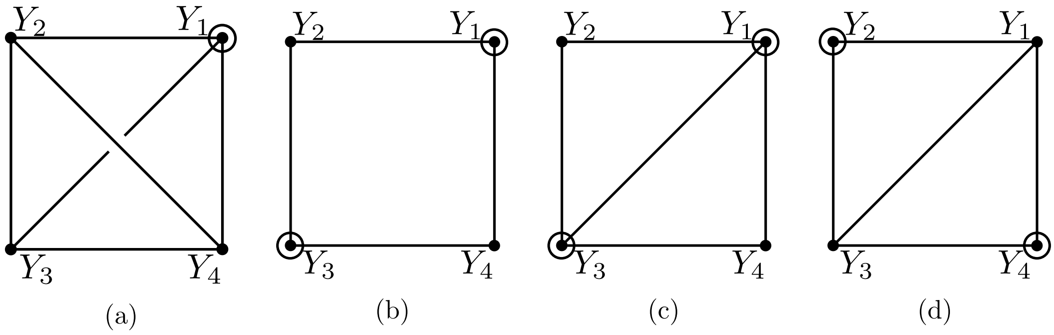



Figure 4.10: Possible dual graphs for Case 4.

in $\mathfrak{X}$. This requires that for any such pair of components, their complement in $S$ is either one or two planar surfaces with at most one boundary component of $S$ contained in each component. In particular, every component of $S \setminus a$ is planar. The marked vertices in Figure 4.10 show components of $S \setminus a$ where it is possible that boundary components could be located (up to symmetries). The condition is that if we remove any edge with its endpoints, there is at most one boundary component of $S$ for each component of the complement. In particular, $S$ can never have more than two boundary components. To ensure that the relevant subsurfaces are planar, it is necessary that if the union of two subsurfaces is connected, the two other subsurfaces meet along at most one curve. Hence, the possibilities for $S$ and $a$ are as shown in Figure 4.11, where any of the boundary components might be filled in with a disc and where the number of curves joining $Y_1$ and $Y_3$ in 4.11c and 4.11d can vary.

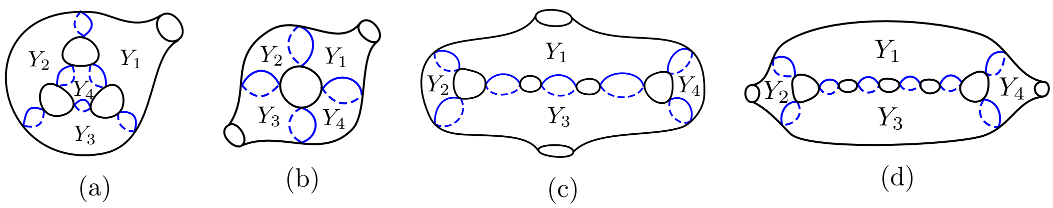



Figure 4.11: Possible surfaces and multicurves for Case 4.

In the case of Figures 4.11b and 4.11c, we have complementary components which are annuli, meaning that two of the curves are isotopic. This should not arise in the multicurve $a$ so we may discard these cases.

Consider Figure 4.11a. Let $\gamma_1$ and $\gamma_4$ be two separating curves in $S \setminus Y_1$, such that $\gamma_1$ has bounded intersection with both $\beta$ and $a$, and $\gamma_4$ has bounded intersection with $\beta'$ and $a$. Apart perhaps from $Y_1$, each component of $S \setminus a$ is a copy of $S_{0,3}$, so $\gamma_1$ and $\gamma_4$ are determined by their arcs of intersection with each component of $S \setminus a$ and by twists around the curves of $a$. Consider a curve $\eta$ in $S \setminus Y_2$ intersecting each of the curves of $a$ joining $Y_1$ and $Y_4$ and joining $Y_3$ and $Y_4$ exactly twice (see Figure 4.12). Up to twists on the boundary of $Y_3$, $\eta \cap Y_3$ consists of one arc intersecting each arc of $\gamma_1 \cap Y_3$ at most twice, and $\eta \cap Y_4$ consists of two arcs intersecting each arc of $\gamma_1 \cap Y_4$ at most once, again up to twists on the boundary of $Y_4$. However, the number of intersections between $\gamma_1$ and $\eta$ is not bounded due to twists around the curve of $a$ joining $Y_3$ and $Y_4$. Twisting $\eta$ appropriately about this curve, retaining the property of being separating, reduces the number of such intersections to below some uniform bound so the intersection number of $\gamma_1$ and the new curve obtained by twisting $\eta$, which we call $\gamma_2$, is bounded in terms of $N$ and $\xi(S)$. Similarly, take a curve $\eta'$ in $S \setminus Y_4$ intersecting exactly twice each of the boundary components of $Y_2$ which meet $Y_1$ or $Y_3$, choosing $\eta'$ to intersect $\eta$ only four times as shown. Twisting $\eta'$ appropriately about the curve of $a$ joining $Y_2$ and $Y_3$, we obtain a separating curve $\gamma_3$ whose intersection number with $\gamma_4$ is bounded in terms of $N$ and $\xi(S)$. Moreover, $i(\gamma_2, \gamma_3) \leqslant 4$. The sequence of curves $\beta, \gamma_1, \gamma_2, \gamma_3, \gamma_4, \beta'$ gives a bound on the distance in $\mathrm{Sep}(S)$ between $\beta$ and $\beta'$ in terms of $N$ and $\xi(S)$.

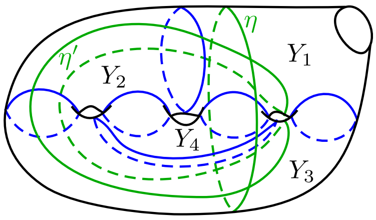



Figure 4.12: The curves used for the case of Figure 4.11a.

Now consider Figure 4.11d. We can assume that both boundary components are present since otherwise at least two of the curves of $a$ shown would be isotopic. Let $\gamma_1$ be a separating curve in $S \setminus Y_1$ with bounded intersection with $\beta$ and $a$, and let $\gamma_2$ be a separating curve in $S \setminus Y_3$ with bounded intersection with $\beta'$ and $a$. These two curves intersect $Y_2$ and $Y_4$ in essential arcs. By the restrictions on which subsurfaces $\gamma_1$ and $\gamma_2$ may intersect, $\gamma_1 \cap Y_2$ is a collection of arcs in $Y_2$ with both endpoints in the boundary component of $Y_2$ which meets $Y_3$. This is a unique isotopy class of arcs in $Y_2$, which is homeomorphic to $S_{0,3}$. Similarly, $\gamma_2 \cap Y_2$ is represented by a unique isotopy class, which intersects the isotopy class of $\gamma_1 \cap Y_2$ twice (see Figure 4.13). The same holds for the intersection of these curves with $Y_4$.

The number of arcs of each of $\gamma_1$ and $\gamma_2$ in $Y_2$ and $Y_4$ is bounded since both curves have bounded intersection with $a$. Hence the number of intersections between $\gamma_1$ and $\gamma_2$ is bounded, so their distance in $\mathrm{Sep}(S)$ is bounded in terms of $N$ and $\xi(S)$.

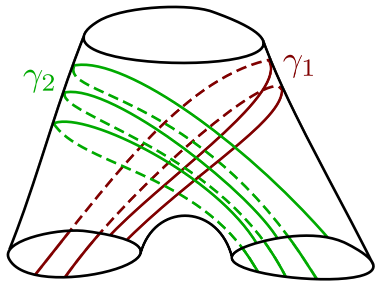



Figure 4.13: An example of the intersection of $\gamma_1$ and $\gamma_2$ with $Y_2$ or $Y_4$ in the case of Figure 4.11d.

**Case 5.** Finally, suppose that $S \setminus a$ has more than four components and that the complement in $S$ of each component is connected. We claim that we can remove curves of $a$ to obtain a vertex of $\mathcal{K}(S)$ which falls into one of the cases above. Firstly, we may assume that removing any single curve from $a$ gives a multicurve which is not a vertex for $\mathcal{K}(S)$ since otherwise we could replace $a$ with a vertex of $\mathcal{K}(S)$ with fewer curves. The dual graph $G$ describing how the components of $S \setminus a$ are connected is a 2-vertex-connected simple graph on at least five vertices. We claim that it is possible to find two edges in $G$ which share no endpoints. Firstly, if $G$ is a complete graph then we may choose any four distinct vertices and there will be a pair of disjoint edges with these as endpoints. Assume now that $G$ is not a complete graph, and let $v$ and $w$ be distinct vertices of $G$ which are not connected by an edge. Let $G'$ be the (not necessarily connected) subgraph of $G$ given by removing $v$, $w$ and all edges containing these vertices. Suppose $v$ is connected by an edge to a single vertex $x$ in $G'$. But then $G \setminus x$ is disconnected, contradicting that $G$ is 2-vertex-connected. Hence $v$ is connected to $G'$ by at least two edges. The same holds for $w$. From the edges joining $v$ to $G'$ and the edges joining $w$ to $G'$, we can choose two edges which are disjoint.

By assumption, removing any curve of $a$ gives a multicurve which is not a vertex of $\mathcal{K}(S)$, and hence which has a complementary component which is in $\mathfrak{X}$. In this way, each edge of $G$ corresponds to a pair of subsurfaces whose union in $S$ is a subsurface in $\mathfrak{X}$, so, in particular, there is a pair of disjoint subsurfaces in $\mathfrak{X}$, $X_1$ and $X_2$, which are each the union of exactly two components of $S \setminus a$. As discussed in Section 4.1.2, $S$ must be either $S_g$, $S_{g,1}$ or $S_{g,2}$, and either $X_2 = S \setminus X_1$ or we are in one other specific case.

First suppose $X_1 = S \setminus X_2$. Then there is no component of $S \setminus a$ which is not contained in either $X_1$ or $X_2$, contradicting that there are at least five components. Now suppose that $X_1 \neq S \setminus X_2$. Then $S = S_{g,1}$ and $X_1$ and $X_2$ are both copies of $S_{0,g+1}$, arranged, up to the action of $\mathrm{MCG}(S)$, as in Figure 4.3b. Let $Y = S \setminus (X_1 \cup X_2)$. Then $Y$ is a copy of $S_{0,3}$, meeting exactly two components of $S \setminus a$, one in $X_1$ and one in $X_2$. Joining $Y$ onto one of these two components gives a subsurface which is still not in $\mathfrak{X}$. Hence, we can remove a curve of $a$ to get a vertex of $\mathcal{K}(S)$ with only four complementary components in $S$. This concludes the final case.

All of the bounds depend only on $N$ and $S$, and we can take the overall bound $N'$ to be the maximum of those found above. Then the diameter of the set $C_a$ is at most $N' = N'(N, \xi(S))$. $\square$

*Proof of Proposition 4.3.1.* Firstly, as already discussed, there exists $N$ such that for any vertex $a$ of $\mathcal{K}(S)$ there exists a separating curve $\gamma$ intersecting $a$ at most $N$ times. Moreover, again as discussed above, the distance between two vertices in $\mathcal{K}(S)$ is bounded above by a function of their intersection number, and so there exists $R = R(N, \xi(S))$ such that $d_{\mathcal{K}(S)}(a, \{\gamma\}) \leqslant R$. Hence, $\phi(\mathrm{Sep}(S))$ is $R$-dense in $\mathcal{K}(S)$.

**Upper bound.** Let $\gamma = \gamma_0, \gamma_1, \ldots, \gamma_{n-1}, \gamma_n = \gamma'$ be a geodesic in $\mathrm{Sep}(S)$. For each $0 \leqslant i \leqslant n-1$, $\gamma_i$ is disjoint from $\gamma_{i+1}$, so $\{\gamma_i, \gamma_{i+1}\}$ is a multicurve, and also necessarily a vertex of $\mathcal{K}(S)$. Hence, $\{\gamma_0\}, \{\gamma_0, \gamma_1\}, \{\gamma_1\}, \ldots, \{\gamma_{n-1}\}, \{\gamma_{n-1}, \gamma_n\}, \{\gamma_n\}$ is a path in $\mathcal{K}(S)$ of length $2n$. Therefore, $d_{\mathcal{K}(S)}(\gamma, \gamma') \leqslant 2d_{\mathrm{Sep}(S)}(\gamma, \gamma')$.

**Lower bound.** Now let us consider the lower bound for the quasi-isometric embedding. Given a vertex $a$ of $\mathcal{K}(S)$, define $C_a = \{\gamma \in \mathrm{Sep}(S) \mid i(\gamma, a) \leqslant N\}$, for sufficiently large $N$, as above. In particular, we can assume $N \geqslant 4$, as will be relevant below. By Lemma 4.3.2, the diameter of $C_a$ is at most $N'$, where $N'$ depends only on $N$ and $\xi(S)$.

Now suppose that $a$ and $b$ are adjacent vertices of $\mathcal{K}(S)$. First assume that the edge joining $a$ and $b$ does not correspond to a move in an $S_{0,4}$ subsurface in $\mathfrak{X}$. Then, without loss of generality, $b$ is obtained from $a$ by adding a single curve. Hence $C_a \cup C_b = C_a$, which has diameter at most $N'$.

Now suppose that $a$ and $b$ differ by a move in an $S_{0,4}$ subsurface. In particular, $S$ is one of $S_3$, $S_{3,1}$, $S_{2,2}$ or $S_{1,3}$. We shall show that $C_a$ and $C_b$ share at least one curve, so the diameter of $C_a \cup C_b$ is at most $2N'$. Let $X$ be the $S_{0,4}$ subsurface in which the move takes place, $\alpha = a \cap X$ and $\beta = b \cap X$.

We first observe that every curve $\gamma$ which is essential and non-peripheral in $X$ separates two boundary components of $X$ on one side and two on the other, and that this partition of $\partial X$ determines whether or not $\gamma$ is separating in $S$. Moreover, if three curves give the vertices of a triangle in the Farey graph $\mathcal{C}(X)$, that is, they pairwise intersect exactly twice, then the three curves give the three different partitions of the components of $\partial X$ into pairs.

Now, suppose first that the subsurface $X$ contains some separating curve of $S$. If $\alpha$ is separating, then $\alpha$ is in $C_a \cap C_b$, since it intersects each of $a$ and $b$ at most twice, and similarly if $\beta$ is separating. If neither $\alpha$ nor $\beta$ is separating then we can take $\gamma$ in $X$ intersecting each of $\alpha$ and $\beta$ exactly twice. Since $\gamma$ will give a different partition of the components of $\partial X$ to $\alpha$ and $\beta$, and since $X$ contains separating curves of $S$, it follows that $\gamma$ must be separating. Once again, $\gamma$ intersects each of $a$ and $b$ at most twice, so $\gamma \in C_a \cap C_b$.

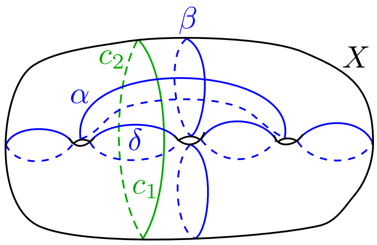



Figure 4.14: An example of how to find a curve in $C_a \cap C_b$, where $a$ and $b$ are connected by a move in an $S_{0,4}$ subsurface which contains no separating curve of $S$.

Suppose now that $X$ does not contain any separating curve of $S$. We claim that given $\alpha$ and $\beta$ in $X \cong S_{0,4}$ intersecting twice, a boundary component $\delta$ of $X$ and a partition of the other three boundary components of $X$ into a set of one and a set of two, we can find an arc $c$ satisfying the following: both endpoints of $c$ are in $\delta$, $c$ separates $X$ into two components which correspond to the chosen partition of the boundary components, and $c$ intersects each of $\alpha$ and $\beta$ at most twice. As above, the curves $\alpha$ and $\beta$ are two vertices of a triangle in the Farey graph $\mathcal{C}(X)$. For the partition we have chosen of three boundary components of $X$, first add the boundary component $\delta$ to the set with one boundary component, to get a partition of all four boundary components of $X$ into two pairs. Now let $\omega$ be either $\alpha$, $\beta$ or the third vertex of a Farey triangle containing these, such that $\omega$ separates the boundary components of $X$ according to this partition. Let $t$ be an arc joining $\delta$ and $\omega$, with the interior of $t$ disjoint from $\alpha$ and $\beta$, and let $c$ be the boundary of a regular neighbourhood of $t \cup \omega$. Then $c$ satisfies the required conditions.

Now, choose a component $\delta$ of $\partial_S X$ and take an essential arc $c_1$ in $S \setminus X$ with both endpoints in $\delta$, choosing $c_1$ to be disjoint from $a \cap (S \setminus X)$. This arc $c_1$ separates the components of $\partial_S X$ in a certain way. We can now choose an arc $c_2$ in $X$ with both endpoints in $\delta$ and separating the same boundary components as $c_1$. We can join up $c_1$ and $c_2$ in such a way that they form a separating curve $\eta$. Moreover, from above, we can choose $c_2$ to intersect each of $\alpha$ and $\beta$ at most twice. See Figure 4.14 for an example of this construction. Hence, $i(\eta, a) \leqslant 4$ and $i(\eta, b) \leqslant 4$. We assume $N \geqslant 4$, so $\eta \in C_a \cap C_b$.

Let $\gamma$ and $\gamma'$ be two separating curves, and $\{\gamma\} = a_0, a_1, \ldots, a_{n-1}, a_n = \{\gamma'\}$ a geodesic in $\mathcal{K}(S)$. For each $1 \leqslant i \leqslant n-1$, choose $\gamma_i$ in $C_{a_i}$, and take $\gamma_0 = \gamma$ and $\gamma_n = \gamma'$. From above, $d_{\mathrm{Sep}(S)}(\gamma_i, \gamma_{i+1}) \leqslant 2N'$ for each $0 \leqslant i \leqslant n-1$. Hence $d_{\mathrm{Sep}(S)}(\gamma, \gamma') \leqslant 2N' d_{\mathcal{K}(S)}(\gamma, \gamma')$. $\square$

By Theorem 4.2.3 and Proposition 3.7.1, this completes the proof of Theorem 4.1.1.

# Chapter 5

# Surgery arguments in the coarse geometry of curve complexes

In this chapter, we give two instances of the use of surgery arguments to obtain uniform constants for coarse geometric properties of graphs associated to surfaces. Section 5.1 gives a new proof of the uniform hyperbolicity of the curve graphs, which is based on methods of Przytycki and Sisto [48]. The proof in [48] applies only for closed surfaces and our proof extends this to also apply for surfaces with boundary. Section 5.2 gives an elementary proof that the disc graphs are uniformly quasiconvex in the curve graphs, also making use of the methods of [48], along with work of Masur and Minsky [43]. The work in this latter section appears in [57].

## 5.1 Uniform hyperbolicity of the curve graphs

In this section, we will give a proof of the uniform hyperbolicity of the curve graphs using surgery arguments. This result is originally due to Aougab [1], Bowditch [12], Clay, Rafi and Schleimer [20] and Hensel, Przytycki and Webb [35], in independent proofs.

The methods of the four proofs are rather different. The arguments we use here are inspired by those of [35]. Hensel, Przytycki and Webb introduce the idea of *unicorn arcs* produced by surgeries to construct paths in the arc graph which are close to geodesics. They prove, using these paths, that arc graphs are uniformly hyperbolic, and deduce from this that curve graphs are uniformly hyperbolic. By analogy with unicorn arcs, Przytycki and Sisto introduce *bicorn curves* in [48] to give a proof that the curve graphs of closed surfaces of genus at least two are uniformly hyperbolic. We here extend the method of [48] to apply also to surfaces with boundary.

In [48], the following criterion for hyperbolicity is used. This appears as Theorem 3.15 of [45] (without the final clause on Hausdorff distance, which we shall use in Section 5.2) and Proposition 3.1 of [12]. This result is also related to work of Gilman [27].

**Proposition 5.1.1.** *Let $\mathcal{G}$ be a connected graph with vertex set $V(\mathcal{G})$ and let $h \geqslant 0$. Let $d_{\mathcal{G}}$ be the combinatorial metric on $\mathcal{G}$. Suppose that for every $x, y \in V(\mathcal{G})$ there is a connected subgraph $\mathcal{L}(x,y) \subseteq \mathcal{G}$, containing $x$ and $y$, satisfying the following:*

1. *for any $x, y \in V(\mathcal{G})$ with $d_{\mathcal{G}}(x,y) \leqslant 1$, the diameter of $\mathcal{L}(x,y)$ in $\mathcal{G}$ is at most $h$;*

2. *for all $x, y, z \in V(\mathcal{G})$, $\mathcal{L}(x,y) \subseteq N_{\mathcal{G}}(\mathcal{L}(x,z) \cup \mathcal{L}(z,y), h)$.*

*Then $\mathcal{G}$ is $\delta$-hyperbolic for some $\delta$ depending only on $h$. Furthermore, there exists $R$ depending only on $h$ such that the Hausdorff distance between $\mathcal{L}(x,y)$ and any geodesic from $x$ to $y$ is at most $R$.* □

We will use the following slight modification.

**Proposition 5.1.2.** *Let $\mathcal{G}$ be a connected graph with vertex set $V(\mathcal{G})$ and let $h, h' \geqslant 0$. Let $d_{\mathcal{G}}$ be the combinatorial metric on $\mathcal{G}$. Suppose that for every $x, y \in V(\mathcal{G})$ there is a (not necessarily connected) subgraph $\mathcal{L}(x,y) \subseteq \mathcal{G}$, containing $x$ and $y$, satisfying the following:*

1. *for any $x, y \in V(\mathcal{G})$, $N_{\mathcal{G}}(\mathcal{L}(x,y), h')$ is connected;*

2. *for any $x, y \in V(\mathcal{G})$ with $d_{\mathcal{G}}(x,y) \leqslant 1$, the diameter of $\mathcal{L}(x,y)$ in $\mathcal{G}$ is at most $h$;*

3. *for all $x, y, z \in V(\mathcal{G})$, $\mathcal{L}(x,y) \subseteq N_{\mathcal{G}}(\mathcal{L}(x,z) \cup \mathcal{L}(z,y), h)$.*

*Then $\mathcal{G}$ is $\delta$-hyperbolic for some $\delta$ depending only on $h$ and $h'$. Furthermore, there exists $R$ depending only on $h$ and $h'$ such that the Hausdorff distance between $\mathcal{L}(x,y)$ and any geodesic from $x$ to $y$ is at most $R$.*

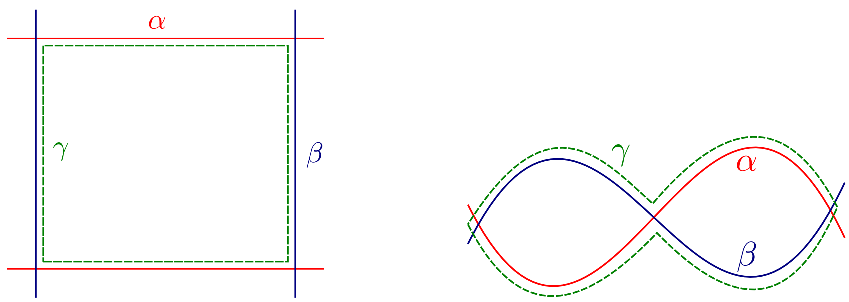
*Proof.* We prove only that Proposition 5.1.2 follows from Proposition 5.1.1. For a proof of Proposition 5.1.1, see [12] or [45]. Suppose subgraphs $\mathcal{L}(x,y)$ satisfy all the hypotheses of this modified proposition, that is, all the hypotheses of Proposition 5.1.1 except with the assumption of connectedness of $\mathcal{L}(x,y)$ replaced by the assumption of coarse connectedness described. Define $\mathcal{L}'(x,y) = N_{\mathcal{G}}(\mathcal{L}(x,y), h')$. This is a connected subgraph of $\mathcal{G}$ containing $x$ and $y$. For any $x, y \in V(\mathcal{G})$

with $d_{\mathcal{G}}(x, y) \leqslant 1$, the diameter of $\mathcal{L}'(x, y)$ in $\mathcal{G}$ is at most $h + 2h'$, and for any $x, y, z \in V(\mathcal{G})$, $\mathcal{L}'(x, y) \subseteq N_{\mathcal{G}}(\mathcal{L}'(x, z) \cup \mathcal{L}'(y, z), h)$. Hence, the conclusion of Proposition 5.1.1 holds, except with constants now depending on $h$ and $h'$. $\square$

Let $S$ be a compact, oriented surface such that $\xi(S) \geqslant 2$. In what follows, we will be working with fixed representatives of isotopy classes of curves, as this will be convenient for the surgeries.

For two essential, non-peripheral curves $\alpha$, $\beta$, fixed in minimal position, we can form new curves by surgeries. We join arcs of $\alpha$ and $\beta$ at points of $\alpha \cap \beta$. To obtain an embedded curve the arcs should not intersect in their interiors. We call curves obtained this way $(\alpha, \beta)$-*curves*. We will allow up to two arcs of each of $\alpha$ and $\beta$, and call the subarcs contained in $\alpha$, $\alpha$-arcs, and the subarcs contained in $\beta$, $\beta$-arcs. We call the points of intersection of the $\alpha$- and $\beta$-arcs *corners*. Note that we will also consider $\alpha$ and $\beta$ to be $(\alpha, \beta)$-curves; in each case the number of corners is zero. We include arrangements of two arcs of each curve where one endpoint is common to all four arcs (see, for example, Figure 5.1b). Here it is necessary to perform an additional surgery in a neighbourhood of this intersection point to get an embedded curve. Following [48], we will call $(\alpha, \beta)$-curves with exactly two corners *bicorn curves* (see Figure 5.2 for examples). We will typically denote arcs of $\alpha$ by $a$, arcs of $\beta$ by $b$ and so on.

**Remark 5.1.3.** In the figures in this section, we show each $(\alpha, \beta)$-curve disjoint from its $\alpha$- and $\beta$-arcs. This is for clarity of the illustrations. We really want to consider the $(\alpha, \beta)$-curve as actually coinciding with its $\alpha$- and $\beta$-arcs (except for in a small neighbourhood if there is an endpoint common to all four arcs).




(a) Four intersection points. (b) Three intersection points.

Figure 5.1: Examples of $(\alpha, \beta)$-curves with four corners.

For $\alpha$ and $\beta$ fixed in minimal position, define $\Lambda(\alpha, \beta)$ to be the set of $(\alpha, \beta)$-curves with at most four corners which are essential and non-peripheral. Let $\mathcal{L}(\alpha, \beta)$

be the full subgraph of $\mathcal{C}(S)$ whose vertices are the isotopy classes of all curves in $\Lambda(\alpha, \beta)$. Note that one vertex in $\mathcal{L}(\alpha, \beta)$ could correspond to more than one representative curve in $\Lambda(\alpha, \beta)$.

Condition 2 of Proposition 5.1.2 is easily verified for $\mathcal{L}(\alpha, \beta)$. If two curves $\alpha$ and $\beta$ are at distance at most 1 in $\mathcal{C}(S)$ then they do not intersect in minimal position, so $\mathcal{L}(\alpha, \beta) = \{\alpha, \beta\}$. Thus $\mathcal{L}(\alpha, \beta)$ has diameter at most 1 in $\mathcal{C}(S)$.

To show hyperbolicity of $\mathcal{C}(S)$, we still need to show the following conditions are satisfied.

**Lemma 5.1.4.** *There exists $h' \geqslant 0$ such that for all $\alpha$, $\beta$, $N_{\mathcal{C}(S)}(\mathcal{L}(\alpha, \beta), h')$ is connected.*

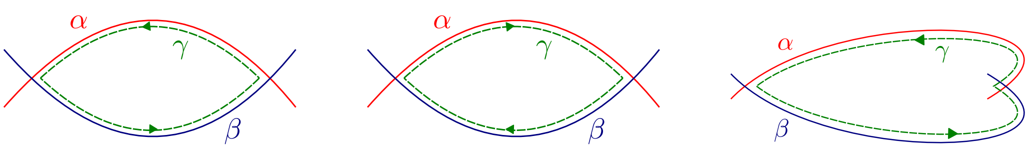
**Lemma 5.1.5.** *There exists $h \geqslant 0$ such that for any three curves $\alpha$, $\beta$, $\delta$ in $S$,*

$$\mathcal{L}(\alpha, \beta) \subseteq N_{\mathcal{C}(S)}(\mathcal{L}(\alpha, \delta) \cup \mathcal{L}(\beta, \delta), h).$$

We begin with the proof of Lemma 5.1.4. Let $\gamma$ be an $(\alpha, \beta)$-curve with at most four corners. The arcs defining $\gamma$ contain two, three or four points of $\alpha \cap \beta$. Orient $\gamma$, and follow an $\alpha$-arc, say, according to this orientation to reach one of these points. To stay on the curve $\gamma$ by following the appropriate $\beta$-arc requires turning either left or right at this intersection point. This gives a sequence of left and right turns associated to each oriented curve, defined up to cyclic permutation. If there are only three points of $\alpha \cap \beta$, then one of them (the point where all four arcs intersect) will be counted twice. We denote left turns by $L$ and right turns by $R$.

The possibilities for an oriented bicorn curve are $LL$, $LR$ and $RR$, as illustrated in Figure 5.2. We will call a curve with sequence $LL$ an $LL$-curve, and so on.

Note that reversing the orientation of the $(\alpha, \beta)$-curve changes left turns to right turns and right turns to left turns. Hence an $LL$-curve is the same as an $RR$-curve with the opposite orientation, and so on.




(a) An $LL$-curve. (b) An $RR$-curve. (c) An $LR$-curve.

Figure 5.2: The possibilities for bicorn curves.

**Lemma 5.1.6.** *An oriented bicorn curve $\gamma$ cannot bound a disc on the left, and $\gamma$ can bound a peripheral annulus on the left only if it is an $LL$-curve.*

*Proof.* Suppose an $LL$-curve $\gamma$ (as in Figure 5.2a) bounds a disc on the left. This is the same as saying that $\alpha$ and $\beta$ form a bigon here, which contradicts the assumption that the two curves are in minimal position.

Now let $\gamma = a \cup b$ be an $LR$-curve (Figure 5.2b) and suppose for contradiction that it bounds a disc $D$ on the left. Since $\gamma$ turns right at one of the two points of $a \cap b$, at this point, $p$, the arcs of $\alpha$ and $\beta$ which are not part of $\gamma$ are on the left, and hence enter the disc $D$. The arc of $\alpha \setminus a$ entering must leave $D$ somewhere, and can do this only by intersecting the boundary of $D$ again. It cannot intersect the component of $\partial D$ which is in $\partial S$ as it is a subarc of a curve, so it must intersect $\gamma$. Moreover, it cannot intersect $a$, so must intersect $b$ at a point $q$. The arc $a'$ of $\alpha \setminus a$ joining $p$ and $q$ in $D$ divides the disc $D$ into two components, each of which must also be a disc. Moreover, each of these components has boundary made up of one arc of $\alpha$ and one arc of $\beta$: for one it is $a' \cup b'$, where $b' \subset b$, and for the other it is $(a \cup a') \cup (b \setminus b')$. Each of these discs is in fact a bigon between $\alpha$ and $\beta$, contradicting minimal position of $\alpha$ and $\beta$.

Similarly, suppose that $\gamma = a \cup b$ is an $LR$-curve and that it it bounds a peripheral annulus $A$ on the left. As above, we find that the arc of $\alpha \setminus a$ crossing $A$ from the right turn divides $A$ into two components, where now one of these components is a disc and the other is an annulus. The disc component is a bigon between $\alpha$ and $\beta$, again contradicting minimal position of $\alpha$ and $\beta$.

The same argument applies for an $RR$-curve (Figure 5.2c), where now there are two points where arcs of $\alpha$ and $\beta$ enter the disc, and there might be an $LR$-curve bounding a disc instead of a bigon, which again we have shown to be a contradiction. $\square$

**Remark 5.1.7.** By reversing the orientation of $\gamma$, we can see that it is equivalent to say that an $LL$-curve can bound some topological type of subsurface on the left and to say that an $RR$-curve can bound the same surface on the right. Similarly, it is equivalent to say that an $LR$-curve can bound a particular subsurface on the left or on the right. Hence Lemma 5.1.6 shows that every bicorn curve is essential, and that every $LR$-curve is also non-peripheral.

**Lemma 5.1.8.** *Let $\gamma$ be an $(\alpha, \beta)$-curve with four corners. Then $\gamma$ can bound a disc on the left only if $\gamma$ is an $LLLL$-curve.*

*Proof.* Suppose $\gamma$ is not an $LLLL$-curve, so that at at least one of the intersection points is a right turn. First suppose that there are four distinct intersection points.

Let $a$ and $b$ be $\alpha$- and $\beta$-arcs respectively, and let $p = a \cap b$ be a right turn. At this intersection point, there is an arc $a'$ of $\alpha$ entering the disc $D$. This arc $a'$ must leave the disc through a point $p'$ in one of the $\beta$-arcs (possibly at a corner), and cuts $D$ into two discs. If $p' \in b$, then $a' \cup b'$ is a bicorn curve, for $b' \subset b$ with endpoints $p$ and $p'$. If $p' \notin b$ then $(a \cup a') \cup b''$ is a bicorn curve, where $b''$ is a subarc of the other $\beta$-arc of $\gamma$. In either case, this gives an inessential bicorn curve, which is a contradiction by Lemma 5.1.6.

Now suppose that there are only three intersection points, as in Figure 5.1b. Let $\gamma$ be made up of $\alpha$-arcs $a_1$ and $a_2$ and $\beta$-arcs $b_1$ and $b_2$, where $a_1$ and $b_1$ intersect at two points and $a_2$ and $b_2$ intersect at two points. Then $\gamma_1 = a_1 \cup b_1$ and $\gamma_2 = a_2 \cup b_2$ are bicorn curves. If $\gamma$ bounds a disc on the left, then $\gamma_1$ and $\gamma_2$ must each also bound a disc, which is again a contradiction by Lemma 5.1.6. Hence, only an $LLLL$-curve can bound a disc on the left. □

**Lemma 5.1.9.** *Any $LRLR$-curve is non-peripheral.*

*Proof.* Let $\gamma$ be an $LRLR$-curve and suppose it has four distinct intersection points. Suppose that $\gamma$ bounds a peripheral annulus $A$ on the left. Let $a$ be an $\alpha$-arc and $b$ a $\beta$-arc in $\gamma$, intersecting at a right turn $p$. An arc $a'$ of $\alpha$ enters the annulus from $p$ and either intersects $\gamma$ again at a point $p'$ in one of the two $\beta$-arcs or meets the other $\alpha$-arc of $\gamma$ at the other right turn. In the first case, it divides $A$ into a disc and an annulus, one of which is bounded by an $LL$-curve and the other by an $LLLR$-curve. In the second case both the disc and annulus components are bounded by $LL$-curves. This is a contradiction in each case since neither an $LL$-curve nor an $LLLR$-curve can bound a disc.

Now suppose that there are only three distinct intersection points in $\gamma$. Then the $\alpha$- and $\beta$-arcs in $\gamma$ pair up to give two $LL$-curves. Since neither of these can bound a disc, $\gamma$ cannot bound a peripheral annulus.

Furthermore, since an $LRLR$-curve cannot bound a peripheral annulus on the left, it also follows, as in Remark 5.1.7, that an $LRLR$-curve cannot bound such an annulus on the right. Hence any $LRLR$-curve is non-peripheral. □

**Lemma 5.1.10.** *Let $\gamma$ be an $(\alpha, \beta)$-curve with at most four corners. Let $k$ be the total number of intersections of $\beta$ with the interiors of the $\alpha$-arcs of $\gamma$. Then $i(\alpha, \beta) \leqslant k + 2$.*

*Proof.* Orient $\gamma$. To take $\gamma$ and $\beta$ into minimal position, we in particular want to ensure that they intersect transversely by isotoping $\gamma$ off its $\beta$-arcs. Take an annular neighbourhood of $\gamma$, and let $\gamma_L$ and $\gamma_R$ be its boundary components on the left and right of $\gamma$ respectively (in the case where one endpoint is common to all four arcs, we choose only the boundary component which does not intersect $\gamma$). Each of these curves is isotopic to $\gamma$. We may arrange that $\gamma_L$ and $\gamma_R$ intersect $\beta$ transversely and that the number of intersections coming from intersections of $\beta$ with the interior of $\alpha$-arcs of $\gamma$ is unchanged. At a left turn, there is an intersection of $\beta$ with $\gamma_R$ coming from this corner, but not with $\gamma_L$. Similarly, at a right turn, there is an intersection of $\beta$ with $\gamma_L$ coming from this corner, but not with $\gamma_R$. Since we can choose whichever of $\gamma_L$ and $\gamma_R$ has smaller intersection with $\beta$ to get an upper bound on $i(\gamma, \beta)$ (except in the one case mentioned above, where we get at most two intersections from the corners), we find that $i(\gamma, \beta)$ is at most two more than the number of intersections of $\beta$ with the interior of $\alpha$-arcs of $\gamma$. $\square$

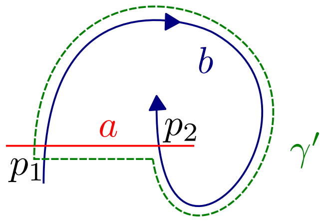
To show that the subgraph $\mathcal{L}(\alpha, \beta)$ is coarsely connected in $\mathcal{C}(S)$, we define a partial order on $\Lambda(\alpha, \beta)$. Recall that the isotopy classes of the curves in $\Lambda(\alpha, \beta)$ make up the vertex set of $\mathcal{L}(\alpha, \beta)$. The partial order is defined by $\gamma' > \gamma$ if the union of the $\alpha$-arcs of $\gamma'$ is strictly contained in the union of the $\alpha$-arcs of $\gamma$, where the $\alpha$-arc of $\beta$ is taken to be empty. In this partial order, $\beta$ is the unique maximal element. Since $\Lambda(\alpha, \beta)$ is finite, a sequence of elements in $\Lambda(\alpha, \beta)$ which is increasing in this partial order eventually terminates in $\beta$. Hence, the following lemma implies that there is a coarse path in $\mathcal{L}(\alpha, \beta)$ connecting each vertex to $\beta$.

**Lemma 5.1.11.** *Let $\gamma \neq \beta$ in $\Lambda(\alpha, \beta)$. There exists $\gamma'$ in $\Lambda(\alpha, \beta)$ such that $\gamma' > \gamma$ and $d_S(\gamma, \gamma') \leqslant 13$.*

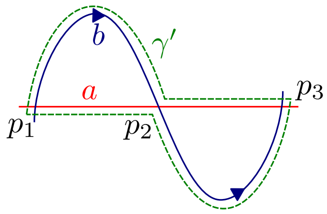
*Proof.* Firstly, if $i(\gamma, \beta) \leqslant 6$ then their distance in $\mathcal{C}(S)$ is at most $2 \cdot 6 + 1 = 13$ (see Lemma 2.1 of [41]), so we can take $\gamma' = \beta$. Suppose $i(\gamma, \beta) \geqslant 7$. By Lemma 5.1.10, there are at least five intersections between $\beta$ and the interior of $\alpha$-arcs of $\gamma$. Since there are at most two $\alpha$-arcs, it follows that $\beta$ must intersect some $\alpha$-arc $a$ of $\gamma$ at least three times outside the endpoints of $\alpha$ (if $\gamma = \alpha$, we can take $a$ to be a subarc of $\alpha$ which contains all intersections with $\beta$).

Let $b$ be an arc of $\beta$ intersecting the interior of $a$ exactly three times. Assume that $b$ is minimal, in that there is no proper subarc of $b$ which has three intersections with the interior of an $\alpha$-arc of $\gamma$. Then $b$ has at most two intersections with another $\alpha$-arc of $\gamma$. There may also be at most two intersections between $b$ and $\gamma$ (once $\gamma$ is slightly isotoped off its $\beta$-arcs) coming from corners of $\gamma$, which may arise if $b$ contains $\beta$-arcs of $\gamma$. We shall form $\gamma' \in \Lambda(\alpha, \beta)$ using subarcs of $a$ and $b$.

Orient $b$ and consider its three intersections with $a$. Either there are two consecutive intersections $p_1$, $p_2$ with the same orientation or the orientations of the intersections alternate. In the first case (see Figure 5.3a) there is an $LR$-curve formed from subarcs of $a$ and $b$ intersecting at $p_1$ and $p_2$. In the second case, either we get an $LR$-curve, as in Figure 5.3a, with an extra intersection with $a$, or we get an $LRLR$-curve, as in Figure 5.3b. Each of these curves must be essential and non-peripheral by the above lemmas. Moreover, from the bound on how many times $b$ can intersect $\gamma$, in each case the resulting curve $\gamma'$ satisfies $i(\gamma, \gamma') \leqslant 6$ and hence $d_S(\gamma, \gamma') \leqslant 13$. $\square$




(a) Consecutive intersections with the same orientation.




(b) Alternating orientations.

Figure 5.3: Examples of finding $\gamma' > \gamma$.

This proves Lemma 5.1.4, where $h'$ can be taken to be 7. Lemma 5.1.5 is implied by the following.

**Lemma 5.1.12.** *Let $\alpha$, $\beta$ and $\delta$ be in minimal position, and let $\gamma$ be a curve in $\Lambda(\alpha, \beta)$. Then there exists $\gamma^*$ in $\Lambda(\alpha, \delta)$ or $\Lambda(\beta, \delta)$ such that $d_S(\gamma, \gamma^*) \leqslant 17$.*

*Proof.* If $\gamma$ and $\delta$ intersect at most eight times then take $\gamma^* = \delta$. Now assume $\gamma$ and $\delta$ intersect at least nine times. Then $\delta$ must intersect some arc in $\gamma$, without loss of generality, an $\alpha$-arc $a$, at least three times. Take an arc $d$ joining three intersections with $a$ consecutive along $\delta$. Assume $d$ to be minimal as with $b$ above, so that any $(\alpha, \delta)$-curve formed from subarcs of $d$ and $a$ intersects each other arc of $\gamma$ at most twice. It is possible to find some such curve $\gamma^*$ which is essential and non-peripheral by the same methods as before, replacing $b$ by $d$. Then $i(\gamma, \gamma^*) \leqslant 8$ and $d_S(\gamma, \gamma^*) \leqslant 17$. $\square$

Therefore, all the conditions of Proposition 5.1.2 are satisfied and this proves that $\mathcal{C}(S)$ is hyperbolic. Moreover, the proofs above did not depend on the surface as long as the complexity was sufficient to have pairs of disjoint curves. Thus, this method shows uniform hyperbolicity.

## 5.2 Uniform quasiconvexity of the disc graphs in the curve graphs

### 5.2.1 Statement of results

When a surface $S$ is a boundary component of a compact, orientable 3-manifold $M$, we can consider the subset of the vertex set of $\mathcal{C}(S)$ which consists of those curves which bound embedded discs in $M$. Equivalently, by Dehn's lemma, these are the essential curves in $S$ which are homotopically trivial in $M$. The *disc graph*, $\mathcal{D}(M, S)$, is the full subgraph spanned by these vertices. Masur and Minsky proved that $\mathcal{D}(M, S)$ is $K$-quasiconvex in $\mathcal{C}(S)$ (see Definition 2.1.4), for some $K$ depending only on the genus of $S$ [43]. The proof relies on a study of nested *train track* sequences. A train track is a graph embedded in a surface which *carries* certain curves, in particular a finite collection of curves called the *vertex cycles* of the train track. More specifically, to any pair of vertices of $\mathcal{D}(M, S)$, Masur and Minsky associate a sequence of curves in $\mathcal{D}(M, S)$, and a nested train track sequence whose vertex cycles are close in $\mathcal{C}(S)$ to the curves of this sequence. They prove that the sets of vertex cycles of nested train track sequences are quasiconvex in $\mathcal{C}(S)$, and the result follows.

This result was improved by Aougab, who showed in [2] that the constants of quasiconvexity for nested train track sequences can be taken to be quadratic in the complexity of the surface, obtaining as a corollary that there exists a function $K(g) = O(g^2)$ such that $\mathcal{D}(M, S)$ is $K(g)$-quasiconvex in $\mathcal{C}(S)$, where $g$ is the genus of $S$. That this bound can be taken to be uniform in the genus of $S$ follows from work of Hamenstädt [29]. In Section 3 of [29], it is shown that the sets of vertex cycles of train track splitting sequences give unparametrised quasigeodesics in $\mathcal{C}(S)$ (that is, they can be reparametrised to give quasigeodesics as in Definition 2.2.1), with constants independent of the surface $S$. Along with the uniform hyperbolicity of the curve graphs, this implies that such subsets are uniformly quasiconvex in $\mathcal{C}(S)$. Here, we give a direct proof of the uniform quasiconvexity of $\mathcal{D}(M, S)$ in $\mathcal{C}(S)$, without using train tracks.

**Theorem 5.2.1.** *There exists $K$ such that, for any compact, orientable 3-manifold $M$ and boundary component $S$ of $M$, the disc graph, $\mathcal{D}(M, S)$, is $K$-quasiconvex in $\mathcal{C}(S)$.*

For the main case, where the genus of $S$ is at least 2, this uses an observation that the disc surgeries of [43] give a path of bicorn curves as defined in [48] (see Section 5.1 for a definition). The lower genus case is straight-forward, and is discussed in Section 5.2.2.

### 5.2.2 Exceptional cases

Let $M$ be a compact, orientable 3-manifold. If a boundary component $S$ has genus at most one then the associated disc graph, $\mathcal{D}(M, S)$, is very simple. Firstly, since there are no essential curves on the sphere, the curve graph of the sphere is empty, so we can ignore any sphere boundary components. We shall see that for a torus boundary component $S$, the graph $\mathcal{D}(M, S)$ contains at most one vertex.

Suppose $S$ is a torus boundary component of the 3-manifold $M$. Suppose an essential curve $\delta$ in $S$ bounds an embedded disc $D$ in $M$. Take a closed regular neighbourhood $N$ of $S \cup D$ in $M$. This is homeomorphic to a solid torus with an open ball removed. Suppose some other curve $\delta'$ in $S$ bounds an embedded disc $D'$ in $M$. We can assume that $D'$ intersects the sphere boundary component of $N$ transversely in simple closed curves. Repeatedly performing surgeries along innermost discs to reduce the number of such curves eventually gives a disc with boundary $\delta'$ which is completely contained in $N$. Therefore, an essential curve in $S$ bounds an embedded disc in $M$ if and only if it bounds an embedded disc in $N$. In $S$, there is, up to isotopy, no curve other than $\delta$ which bounds an embedded disc in $N$, since such a curve must be trivial in $H_1(N; \mathbb{Z})$. We hence find that if $S$ is any torus boundary component, then $\mathcal{D}(M, S)$ is at most a single point. In this case, $\mathcal{D}(M, S)$ is 0-quasiconvex, or convex, in the curve graph of $S$ (which is the Farey graph, as described in Section 3.3.1).

### 5.2.3 Proof of the main result

Now let $S$ be a boundary component of genus at least 2 of a compact, orientable 3-manifold $M$, and $\mathcal{D}(M, S)$ the associated disc graph. To prove that $\mathcal{D}(M, S)$ is uniformly quasiconvex in $\mathcal{C}(S)$, we will again make use of Proposition 5.1.2. This time the important result will be the final clause on Hausdorff distances. As noted above, Proposition 5.1.2 is a slight adaptation of Proposition 5.1.1, which appears in [12] and [45].

Given two curves $\alpha$ and $\beta$ in $S$, we shall define $\Theta(\alpha, \beta)$ to be the subgraph of $\mathcal{C}(S)$ containing the isotopy classes of $\alpha$, $\beta$ and all bicorn curves between $\alpha$ and $\beta$. We could just as well take the set of curves $\Lambda(\alpha, \beta)$ (or their isotopy classes) from Section 5.1 and use results from this section. However, since bicorn curves are all we shall need, we shall instead quote results from [48].

Przytycki and Sisto define in [48] an "augmented curve graph", $\mathcal{C}_{aug}(S)$, where two curves are adjacent if they intersect at most twice. Such curves cannot fill $S$ (which has genus at least 2) so are at distance at most 2 in $\mathcal{C}(S)$. Given two curves $\alpha$ and $\beta$ in minimal position, $\eta(\alpha,\beta)$ is defined in [48] to be the full subgraph of $\mathcal{C}_{aug}(S)$ spanned by $\Theta(\alpha,\beta)$. This is shown to be connected for all $\alpha$ and $\beta$. It is further verified that the hypotheses of Proposition 5.1.1 are satisfied when $\mathcal{G}$ is $\mathcal{C}_{aug}(S)$, $\mathcal{L}(\alpha,\beta)$ is $\eta(\alpha,\beta)$ for each $\alpha$, $\beta$, and $h$ is 1, independently of the surface $S$.

Since $\eta(\alpha,\beta)$ is connected in $\mathcal{C}_{aug}(S)$, for any $\gamma,\gamma' \in \Theta(\alpha,\beta)$, there is a sequence $\gamma=\gamma_0,\gamma_1,\dots,\gamma_n=\gamma'$ of curves in $\Theta(\alpha,\beta)$, where $d_S(\gamma_{i-1},\gamma_i)\leqslant 2$ for each $1\leqslant i\leqslant n$. Hence, $N_{\mathcal{C}(S)}(\Theta(\alpha,\beta),1)$ is a connected subgraph of $\mathcal{C}(S)$. Moreover, if $d_S(\alpha,\beta)\leqslant 1$, then $\alpha$ and $\beta$ are disjoint, so $\Theta(\alpha,\beta)$ contains no other curves and its diameter in $\mathcal{C}(S)$ is at most 1. Finally, since $\eta(\alpha,\beta)\subset N_{\mathcal{C}_{aug}(S)}(\eta(\alpha,\delta)\cup\eta(\beta,\delta),1)$ for any curves $\alpha$, $\beta$, $\delta$, we have $\Theta(\alpha,\beta)\subset N_{\mathcal{C}(S)}(\Theta(\alpha,\delta)\cup\Theta(\beta,\delta),2)$. Now using Proposition 5.1.2, this proves the following lemma.

**Lemma 5.2.2.** *There exists $R$ such that, for any closed, orientable surface $S$ of genus at least 2, and any curves $\alpha$, $\beta$ in $S$, the Hausdorff distance in $\mathcal{C}(S)$ between $\Theta(\alpha,\beta)$ and any geodesic in $\mathcal{C}(S)$ joining $\alpha$ and $\beta$ is at most $R$.*

We now show that, moreover, any geodesic between $\alpha$ and $\beta$ in $\mathcal{C}(S)$ lies in a uniform neighbourhood of any path within $\Theta(\alpha,\beta)$ connecting $\alpha$ and $\beta$.

**Lemma 5.2.3.** *Let $\alpha$, $\beta$ be two curves in $S$, $P(\alpha,\beta)$ a path from $\alpha$ to $\beta$ in $\mathcal{C}(S)$ with all vertices in $\Theta(\alpha,\beta)$, and $g$ a geodesic in $\mathcal{C}(S)$ joining $\alpha$ and $\beta$. Then $g$ is contained in the $(2R+2)$-neighbourhood of $P(\alpha,\beta)$.*

*Proof.* This uses a well known connectedness argument. From Lemma 5.2.2, $P(\alpha,\beta)$ is contained in $N_{\mathcal{C}(S)}(g,R)$. Take any vertex $\gamma$ in $g$. Let $g_0$ be the subpath of $g$ from $\alpha$ to $\gamma$ and $g_1$ the subpath from $\gamma$ to $\beta$. Then the three sets $N_{\mathcal{C}(S)}(g_0,R+1)$, $N_{\mathcal{C}(S)}(g_1,R+1)$ and $P(\alpha,\beta)$ intersect in at least one vertex, say $\delta$. Let $\gamma_0$ in $g_0$ and $\gamma_1$ in $g_1$ be such that $d_S(\gamma_0,\delta)\leqslant R+1$ and $d_S(\gamma_1,\delta)\leqslant R+1$. Now $d_S(\gamma_0,\gamma_1)\leqslant 2R+2$ and $\gamma$ is in the (geodesic) subpath of $g$ from $\gamma_0$ to $\gamma_1$, so $d_S(\gamma,\gamma_i)\leqslant R+1$ for either $i=0$ or $i=1$. Hence, $d_S(\gamma,\delta)\leqslant 2R+2$. Since $\gamma$ was an arbitrary vertex in $g$ and $\delta$ is in $P(\alpha,\beta)$, we have $g\subset N_{\mathcal{C}(S)}(P(\alpha,\beta),2R+2)$. □

Given that $\alpha$ and $\beta$ bound embedded discs in $M$, we now describe how to choose $P(\alpha,\beta)$ so that all curves in the path are also vertices of $\mathcal{D}(M,S)$, following Section 2 of [43].

Assume curves $\alpha$ and $\beta$ are fixed in minimal position and choose a subarc $J\subset\alpha$. Masur and Minsky define several curve replacements, of which we shall need only the following. A *wave curve replacement* with respect to $(\alpha,\beta,J)$ is the replacement of $\alpha$ and $J$ by $\alpha_1$ and $J_1$ as follows (see Figure 5.4). Let $w$ be a subarc of $\beta$ with interior disjoint from $\alpha$, and endpoints $p$, $q$ in the interior of $J$. Suppose that $w$ meets the same side of $J$ at both $p$ and $q$; then $w$ is called a *wave*. Let $J_1$ be the (proper) subarc of $J$ with endpoints $p$, $q$, and define $\alpha_1$ to be the curve $w \cup J_1$. This is an essential curve since $\alpha$ and $\beta$ are in minimal position, so, in particular, no subarc of $J$ and subarc of $\beta$ can form a bigon. Where $\text{int}(J) \cap \beta = \varnothing$, we define a curve replacement with respect to $(\alpha, \beta, J)$ by $\alpha_1 = \beta$, $J_1 = \varnothing$.

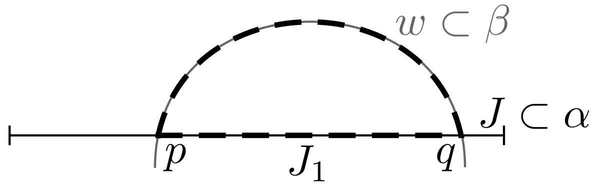



Figure 5.4: A wave curve replacement. The dashed curve is $\alpha_1$.

**Remark 5.2.4.** In [43], it is arranged that $\alpha_1$ and $\beta$ must intersect transversely and be in minimal position by requiring an additional condition on the wave $w$ and by slightly isotoping $w \cup J_1$ to be disjoint from $w$. However, this will not be necessary here, so we choose to simplify the exposition by removing this condition.

Notice that since $\alpha$ does not intersect $\text{int}(w)$, $i(\alpha, \alpha_1) = 0$. Moreover, $\alpha_1 \cap \beta$ consists of the arc $w$ and a set of points which are all contained in the interior of $J_1$, and $|\beta \cap \text{int}(J_1)| < |\beta \cap \text{int}(J)|$ whenever $|\beta \cap \text{int}(J)|$ is non-zero.

We can iterate this process as follows. Athough $\alpha_1$ and $\beta$ coincide in an arc, any intersections in the interior of the subarc $J_1$ are still transverse. Moreover, no subarc of $J_1$ can form a bigon with a subarc of $\beta$, since $\alpha$ and $\beta$ are in minimal position. Hence, we may still define a wave curve replacement with respect to $(\alpha_1, \beta, J_1)$ as for $(\alpha, \beta, J)$ above and obtain an essential curve. A *nested curve replacement sequence* is a sequence $\{(\alpha_i, J_i)\}$ of curves $\alpha = \alpha_0, \alpha_1, \ldots, \alpha_n$ and subarcs $\alpha \supset J_0 \supset J_1 \supset \cdots \supset J_n$, such that $J_0$ contains all points of $\alpha \cap \beta$ in its interior, and such that $\alpha_{i+1}$ and $J_{i+1}$ are obtained by a curve replacement with respect to $(\alpha_i, \beta, J_i)$. We will allow only wave curve replacements in the sequence and not the other curve replacements possible in [43]. We always have $i(\alpha_i, \alpha_{i+1}) = 0$, as for $\alpha$ and $\alpha_1$. Observe that all curves $\alpha_i$ in this sequence are bicorn curves between $\alpha$ and $\beta$, since the nested arcs $J_i$ ensure that they are formed from exactly one arc of $\alpha$ and one of $\beta$.

The following is a case of Proposition 2.1 of [43]. We include a proof for completeness, with the minor modification of the slightly different curve replacements.

**Proposition 5.2.5.** *Let $S$ be a boundary component of a compact, orientable 3-manifold $M$, and let $\alpha$ and $\beta$ be two curves in $S$ in minimal position, each of which bounds an embedded disc in $M$. Let $J_0 \subset \alpha$ be a subarc containing all points of $\alpha \cap \beta$ in its interior. Then there exists a nested curve replacement sequence $\{(\alpha_i, J_i)\}$, with $\alpha_0 = \alpha$, such that:*

- *each $\alpha_i$ bounds an embedded disc in $M$,*
- *the sequence terminates with $\alpha_n = \beta$.*

*Proof.* Suppose that $\alpha$ and $\beta$ bound properly embedded discs $A$ and $B$ respectively. We can assume that $A$ and $B$ intersect transversely, so their intersection locus is a collection of properly embedded arcs and curves. Furthermore, we can remove any curve components by repeatedly performing surgeries along innermost discs, so that $A$ and $B$ intersect only in properly embedded arcs. We will perform surgeries on these discs to get a sequence of discs $A_i$ with $\partial A_i = \alpha_i$. Throughout the surgeries, we will keep $A$ and $B$ fixed, and each $A_i$, except $A_0 = A$ and $A_n = B$, will be a union of exactly one subdisc of each of $A$ and $B$.

Suppose the sequence is constructed up to $\alpha_i = \partial A_i$. If $\beta \cap \alpha_i$ is empty, then $\alpha_{i+1} = \beta = \partial B$ by definition, so the sequence is finished.

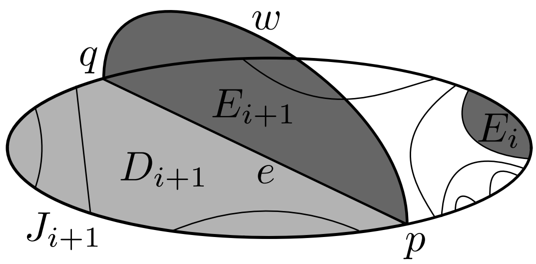



Figure 5.5: The disc surgeries of Proposition 5.2.5. The horizontal disc is $A_i$, shown with arcs of intersection with $B$.

Suppose $\beta$ intersects $\alpha_i$ (as illustrated in the example of Figure 5.5). Let $A_i = D_i \cup E_i$, where $D_i$ is a subdisc of $A$ and $E_i$ is a subdisc of $B$. If $i = 0$, then $E_i$ is empty. If $i > 0$, let $J_i$ be the arc of $\partial D_i$ which is contained in $\partial A_i$. Any point of intersection of $\beta$ and $J_i$ is an endpoint of an arc of intersection of $B$ and $D_i$. Let $E_{i+1}$ be an outermost component of $B \setminus (A_i \cap B)$. Then $E_{i+1}$ is a disc in $B$ such that the boundary of $E_{i+1}$ is made up of an arc $e$ in $\mathrm{int}(D_i) \cap B$ and a subarc $w$ of $\beta$, and such that the interior of $E_{i+1}$ is disjoint from $A_i$. This in particular means that the interior of $w$ is disjoint from $\alpha_i$, that the endpoints $p$, $q$ of $w$ lie in the interior of $J_i$, and that $w$ meets the same side of $J_i$ at both of these endpoints, so $w$ is a wave. Let

$J_{i+1}$ be the subarc of $J_i$ with endpoints $p$, $q$. Let $D_{i+1}$ be the disc in $A_i$ bounded by $e \cup J_{i+1}$. This disc is contained in $D_i$ and hence in $A$. The curve $w \cup J_{i+1}$, with interval $J_{i+1}$, is the wave curve replacement $\alpha_{i+1}$ obtained from $(\alpha_i, \beta, J_i)$, and it is also the boundary of the embedded disc $A_{i+1} = D_{i+1} \cup E_{i+1}$.

Since at each stage $|\beta \cap \text{int}(J_i)|$ decreases, this terminates with $|\beta \cap \text{int}(J_{n-1})| = 0$ and $\alpha_n = \beta$. □

This sequence defines the vertices of a path $P(\alpha, \beta)$ in $\mathcal{C}(S)$, with these vertices contained in both $\mathcal{D}(M, S)$ and $\Theta(\alpha, \beta)$. By Lemma 5.2.3, there exists $K$, independent of $S$, $\alpha$ and $\beta$, such that any geodesic $g$ joining $\alpha$ and $\beta$ in $\mathcal{C}(S)$ is contained in the closed $K$-neighbourhood of $P(\alpha, \beta)$. Hence, $g$ is contained in the closed $K$-neighbourhood of $\mathcal{D}(M, S)$, completing the proof of Theorem 5.2.1.

# Bibliography


[1] Tarik Aougab, *Uniform hyperbolicity of the graphs of curves* : Geom. Topol. **17** (2013) 2855–2875

[2] Tarik Aougab, *Quadratic bounds on the quasiconvexity of nested train track sequences* : Topology Proc. **44** (2014) 365–388

[3] Javier Aramayona, Ferrán Valdez, *On the geometry of graphs associated to infinite-type surfaces* : to appear in Math. Z.

[4] Hans-J. Bandelt, Jarmila Hedlíková, *Median algebras* : Discrete Math. **45** (1983) 1–30

[5] Jason A. Behrstock, *Asymptotic geometry of the mapping class group and Teichmüller space* : Geom. Topol. **10** (2006) 2001–2056

[6] Jason A. Behrstock, Mark F. Hagen, Alessandro Sisto, *Hierarchically hyperbolic spaces I: curve complexes for cubical groups* : Geom. Topol. **21** (2017) 1731–1804

[7] Jason A. Behrstock, Mark F. Hagen, Alessandro Sisto, *Hierarchically hyperbolic spaces II: combination theorems and the distance formula* : preprint (2015), available at `arXiv:1509.00632`

[8] Jason A. Behrstock, Mark F. Hagen, Alessandro Sisto, *Quasiflats in hierarchically hyperbolic spaces* : preprint (2017), available at `arXiv:1704.04271`

[9] Jason A. Behrstock, Yair N. Minsky, *Centroids and the rapid decay property in mapping class groups* : J. London Math. Soc. **84** (2011) 765–784

[10] Brian H. Bowditch, *A course on geometric group theory* : MSJ Mem. **16**, Math. Soc. Japan (2006)

[11] Brian H. Bowditch, *Coarse median spaces and groups* : Pac. J. Math. **261** (2013) 53–93

[12] Brian H. Bowditch, *Uniform hyperbolicity of the curve graphs* : Pac. J. Math. **269** (2014) 269–280

[13] Brian H. Bowditch, *Large-scale rigidity properties of the mapping class groups* : Pac. J. Math. **293** (2018) 1–73

[14] Brian H. Bowditch, *The ending lamination theorem* : preprint (2016), available at `http://homepages.warwick.ac.uk/~masgak/papers/elt.pdf`

[15] Tara E. Brendle, Dan Margalit, *Commensurations of the Johnson Kernel* : Geom. Topol. **8** (2004) 1361–1384

[16] Tara E. Brendle, Dan Margalit, *Normal subgroups of mapping class groups and the metaconjecture of Ivanov* : preprint (2017), available at `arXiv:1710.08929`

[17] Martin R. Bridson, André Haefliger, *Metric spaces of non-positive curvature* : Springer-Verlag Berlin Heidelberg (1999)

[18] Jeffrey F. Brock, *The Weil–Petersson metric and volumes of 3-dimensional hyperbolic convex cores* : J. Am. Math. Soc. **16** (2003) 495–535

[19] Jeffrey F. Brock, Richard D. Canary, Yair N. Minsky, *The classification of Kleinian surface groups II: The Ending Lamination Conjecture* : Ann. Math. **176** (2012) 1–149

[20] Matt Clay, Kasra Rafi, Saul Schleimer, *Uniform hyperbolicity of the curve graph via surgery sequences* : Algebr. Geom. Topol **14** (2014) 3325–3344

[21] Michel Coornaert, Thomas Delzant, Athanase Papadopoulos, *Géométrie et théorie des groupes: Les groupes hyperboliques de Gromov* : Lecture Notes in Math. **1441** Springer-Verlag Berlin Heidelberg (1990)

[22] Matthew Gentry Durham, Federica Fanoni, Nicholas G. Vlamis, *Graphs of curves on infinite-type surfaces with mapping class group actions* : to appear in Ann. Inst. Fourier

[23] Clifford J. Earle, Irwin Kra, *On isometries between Teichmüller spaces* : Duke Math. J. **41** 583–591

[24] David B. A. Epstein, *Curves on 2-manifolds and isotopies* : Acta Math. **115** (1966) 83–107

[25] Benson Farb, Nikolai V. Ivanov, *The Torelli geometry and its applications* : research announcement, Math. Res. Lett. **12** (2005) 293–301

[26] Benson Farb, Dan Margalit, *A Primer on Mapping Class Groups* : Princeton Univ. Press, Princeton (2012)

[27] Robert H. Gilman, *On the definition of word hyperbolic groups* : Math. Z. **242** (2002) 529–541

[28] Mikhail Gromov, *Hyperbolic Groups* : in “Essays in group theory”, ed. S. M. Gersten, Math. Sci. Res. Inst. Publ. **8** Springer, New York (1987) 75–263

[29] Ursula Hamenstädt, *Geometry of graphs of discs in a handlebody: Surgery and intersection* : preprint (2014), available at `arXiv:1101.1843v4`

[30] John L. Harer, *Stability of the homology of the mapping class groups of orientable surfaces* : Ann. Math. **121** (1985) 215–249

[31] John L. Harer, *The virtual cohomological dimension of the mapping class group of an orientable surface* : lnvent. Math. **84** (1986) 157–176

[32] William J. Harvey, *Boundary structure of the modular group* : in “Riemann surfaces and related topics: Proceedings of the 1978 Stony Brook conference”, ed. I. Kra, B. Maskit, Ann. of Math. Stud. **97**, Princeton Univ. Press (1981) 245–251

[33] Allen Hatcher, *Pants Decompositions of Surfaces* : preprint (1999), available at `arXiv:math/9906084`

[34] John Hempel, *3-Manifolds as viewed from the curve complex* : Topology **40** (2001) 631–657

[35] Sebastian Hensel, Piotr Przytycki, Richard C. H. Webb, *1-slim triangles and uniform hyperbolicity for arc graphs and curve graphs* : J. Eur. Math. Soc. **17** (2015) 755–762

[36] Nikolai V. Ivanov, *Automorphisms of complexes of curves and of Teichmüller spaces* : Int. Math. Res. Notices **14** (1997) 651–666

[37] Eduard Looijenga, *Connectivity of complexes of separating curves* : Groups Geom. Dyn. **7** (2013) 443–450

[38] Linda Keen, *Collars on Riemann Surfaces* : in “Discontinuous Groups and Riemann Surfaces: Proceedings of the 1973 Conference at the University of Maryland”, ed. L. Greenberg, Ann. of Math. Stud. **79**, Princeton Univ. Press (1974) 263–268

[39] John D. McCarthy, William R. Vautaw, *Automorphisms of Torelli groups* : preprint (2003), available at `arXiv:math/0311250`

[40] Johanna Mangahas, *A recipe for short-word pseudo-Anosovs* : Amer. J. Math. **135** (2013) 1087–1116

[41] Howard A. Masur, Yair N. Minsky, *Geometry of the complex of curves I: Hyperbolicity* : Invent. Math. **138** (1999) 103–149

[42] Howard A. Masur, Yair N. Minsky, *Geometry of the complex of curves II: Hierarchical structure* : Geom. Funct. Anal. **10** (2000) 902–974

[43] Howard A. Masur, Yair N. Minsky, *Quasiconvexity in the curve complex* : in "In the tradition of Ahlfors and Bers, III", ed. W. Abikoff, A. Haas, Contemp. Math. **355**, Amer. Math. Soc. (2004) 309–320

[44] Howard A. Masur, Saul Schleimer, *The pants complex has only one end* : in "Spaces of Kleinian Groups", ed. Y. Minsky, M. Sakuma, C. Series, Lond. Math. Soc. Lec. Notes **329**, Cambridge Univ. Press (2006) 209–218

[45] Howard A. Masur, Saul Schleimer, *The geometry of the disk complex* : J. Amer. Math. Soc. **26** (2013) 1–62

[46] Yair N. Minsky, *The classification of Kleinian surface groups, I: models and bounds* : Ann. Math. **171** (2010) 1–107

[47] Graham A. Niblo, Nick Wright, Jiawen Zhang, *A four point characterisation for coarse median spaces* : preprint (2017), available at `arXiv:1708.06960`

[48] Piotr Przytycki, Alessandro Sisto, *A note on acylindrical hyperbolicity of mapping class groups* : "Hyperbolic geometry and geometric group theory", ed. K. Fujiwara, S. Kojima, K. Ohshika, Adv. Stud. Pure Math. **73**, Math. Soc. Japan (2017) 255–264

[49] Andrew Putman, *A note on the connectivity of certain complexes associated to surfaces* : Enseign. Math. **54** (2008) 287–301

[50] Alexander J. Rasmussen, *Uniform hyperbolicity of the graphs of nonseparating curves via bicorn curves* : preprint (2017), available at `arXiv:1707.08283`

[51] Martin Roller, *Poc sets, median algebras and group actions: An extended study of Dunwoody's construction and Sageev's theorem* : habilitationschrift, Regensburg (1998), available at `arXiv:1607.07747`

[52] Halsey L. Royden, *Automorphisms and isometries of Teichmüller space* : in "Advances in the theory of Riemann surfaces", ed. L. Ahlfors, L. Bers, H. Farkas, R. Gunning, I. Kra, H. Rauch, Ann. of Math. Stud. **66**, Princeton Univ. Press (1971) 369–384

[53] Michah Sageev, *CAT*(0) *Cube Complexes and Groups* : in "Geometric Group Theory", ed. M. Bestvina, M. Sageev, K. Vogtmann, IAS/Park City Mathematics Series **21** (2014) 7–54

[54] Saul Schleimer, *Notes on the complex of curves* : unpublished notes (2006), available at `http://homepages.warwick.ac.uk/~masgar/Maths/notes.pdf`

[55] Alessandro Sisto, *What is a hierarchically hyperbolic space?* : to appear in "Beyond Hyperbolicity", ed. M. Hagen, R. Webb, H. Wilton, Cambridge Univ. Press

[56] Harold Mark Sultan, *The Asymptotic Cone of Teichmuller Space: Thickness and Divergence* : PhD thesis, Columbia University (2012)

[57] Kate M. Vokes, *Uniform quasiconvexity of the disc graphs in the curve graphs* : to appear in "Beyond Hyperbolicity", ed. M. Hagen, R. Webb, H. Wilton, Cambridge Univ. Press

[58] Kate M. Vokes, *Hierarchical hyperbolicity of the separating curve graph* : preprint (2017), available at `arXiv:1711.03080`

[59] Richard C. H. Webb, *Uniform bounds for bounded geodesic image theorems* : J. Reine Angew. Math. **709** (2015) 219–228